

## **HYDRODYNAMICS FOR AN IDEAL FLUID: HAMILTONIAN FORMALISM AND LIOUVILLE-EQUATION**

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A Hamiltonian formalism for hydrodynamics of ideal fluids is developed with the help of Seliger and Whitham's variational principle. It is shown that a density distribution function in the phase space of the mass-density, momentum-density and energy-density fields obeys a Liouville-equation.

### **1. Introduction**

Clebsch<sup>1,2)</sup> was the first to derive, in 1859, the hydrodynamic equations for an ideal fluid from a variational principle for Euler coordinates. His derivation was, however, restricted to the case of an incompressible fluid. Later Bateman<sup>3)</sup> showed that the analysis of Clebsch also applies to compressible fluids if the pressure is a function of density alone. Finally, in 1968, Seliger and Whitham<sup>4)</sup> formulated a Lagrangian density for the most general case, i.e. taking also into account the dependence on entropy. From a Lagrangian formalism it is of course in general possible to go over to a Hamiltonian description. For hydrodynamics, this was done by Kronig and Thellung<sup>5,6)</sup> in order to quantize the fluid equations. As they based their work on Bateman's analysis, their results only apply to the case of isentropic (or, alternatively, isothermal) flow.

Recently, there has been renewed interest in a Hamiltonian formulation of hydrodynamics. In an interesting paper Enz and Turski<sup>7)</sup> considered hydrodynamic fluctuations on the basis, and with the limitations, of the formalism developed by Thellung<sup>6)</sup>.

In this paper we will develop a Hamiltonian formalism for the general case of Seliger and Whitham and discuss a number of statistical properties of an ideal fluid. This discussion will enable us to study in a subsequent paper nonlinear fluctuations in a real fluid.

In section 2 we discuss the Clebsch representation of the fluid velocity field. Seliger and Whitham's<sup>4)</sup> variational principle, which is based on this representation, is reviewed in section 3. We then introduce a Hamiltonian description of hydrodynamics and define Poisson-brackets in terms of the

canonical (nonphysical) fields in a standard way. From these one may derive Poisson-bracket expressions for the physical (hydrodynamic) fields. These turn out to be identical to those introduced on the basis of symmetry considerations and without the use of canonical fields by Dzyaloshinskii and Volovick<sup>8</sup>).

In section 4 we consider an ensemble of ideal fluids and study the evolution in time of the density distribution function in the "phase space" of physical fields. We conclude that if the physical fields are the mass-density, momentum-density and energy-density, the flow in the corresponding phase space is incompressible, so that the density distribution function obeys a Liouville-equation. The standard equilibrium distributions of fluctuation theory are of course stationary solutions of this equation. This is discussed in section 5.

## 2. The Clebsch representation for an ideal fluid

The behaviour of an ideal fluid is described by the five hydrodynamic equations (conservation laws)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho v, \quad (2.1)$$

$$\frac{\partial \rho v}{\partial t} = -\nabla \cdot \rho v v - \nabla p, \quad (2.2)$$

$$\frac{\partial \rho s}{\partial t} = -\nabla \cdot \rho v s, \quad (2.3)$$

where  $\rho(r, t)$  is the mass density,  $v(r, t)$  the velocity,  $p(r, t)$  the hydrostatic pressure and  $s(r, t)$  the entropy per unit of mass. These equations can equivalently be written as

$$\rho \frac{dv}{dt} = \nabla \cdot v, \quad (2.4)$$

$$\rho \frac{dv}{dt} = -\nabla p, \quad (2.5)$$

$$\rho \frac{ds}{dt} = 0, \quad (2.6)$$

where  $v \equiv \rho^{-1}$  is the specific volume, and where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v \cdot \nabla \quad (2.7)$$

is the total time derivative.

In order to formulate a variational principle, as will be done in the next section, it is convenient to introduce the Clebsch representation<sup>1,2)</sup>

$$\mathbf{v} = -\nabla\phi_p - \lambda\nabla\phi_\lambda - s\nabla\phi_s, \quad (2.8)$$

where the three components of the velocity field are given in terms of four scalar fields as well as the entropy\*. Clearly one of the fields is redundant. In fact, one could, in principle, always represent an arbitrary velocity field with the choice  $\phi_s = 0$ . As will become apparent below, however, it is convenient to choose  $\phi_s$  unequal to zero if the flow is not isentropic (i.e.  $\nabla s \neq 0$ ). In this case it will be necessary that an additional equation specifies this field.

In the Clebsch representation, the vorticity of the fluid is given by

$$\boldsymbol{\omega} \equiv \nabla \wedge \mathbf{v} = -\nabla\lambda \wedge \nabla\phi_\lambda - \nabla s \wedge \nabla\phi_s. \quad (2.9)$$

The total time derivative of the velocity field may then be written as

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 - \mathbf{v} \wedge \boldsymbol{\omega} \\ &= -\nabla\left(\dot{\phi}_p + \lambda\dot{\phi}_\lambda + s\dot{\phi}_s - \frac{1}{2}v^2\right) + \left(\frac{d\phi_\lambda}{dt}\right)\nabla\lambda - \left(\frac{d\lambda}{dt}\right)\nabla\phi_\lambda + \left(\frac{d\phi_s}{dt}\right)\nabla s, \end{aligned} \quad (2.10)$$

where partial time derivatives are denoted by a dot. In obtaining this equation, use has been made of eq. (2.6). On the other hand, eq. (2.5) may be written in the form

$$\frac{d\mathbf{v}}{dt} = -\nabla h + T\nabla s, \quad (2.11)$$

where  $h$  is the enthalpy per unit of mass and  $T$  the temperature, and where we have used the thermodynamic relation

$$\frac{1}{\rho} dp = dh - T ds. \quad (2.12)$$

The four scalar fields  $\phi_p$ ,  $\lambda$ ,  $\phi_\lambda$  and  $\phi_s$  must therefore be chosen in such a way that the right-hand side of eq. (2.10) reduces to the right-hand side of eq. (2.11). This requirement is not sufficient to determine uniquely the equations satisfied by these fields. An appropriate and customary choice<sup>4)</sup> is

$$\dot{\phi}_p + \lambda\dot{\phi}_\lambda + s\dot{\phi}_s - \frac{1}{2}v^2 = h, \quad (2.13)$$

$$\frac{d\lambda}{dt} = 0, \quad (2.14)$$

\* For the special case of isentropic flow, the last term in eq. (2.8) may be absorbed into the first, so that  $\mathbf{v} = -\nabla\phi - \lambda\nabla\phi_\lambda$  where  $\phi = \phi_p + s\phi_s$ .

$$\frac{d\phi_\lambda}{dt} = 0, \quad (2.15)$$

$$\frac{d\phi_s}{dt} = T. \quad (2.16)$$

The last equation may be viewed as the equation necessary to specify the redundant field.

The equations of motion for the six fields ( $\rho$ ,  $s$ ,  $\lambda$ ,  $\phi_\rho$ ,  $\phi_s$  and  $\phi_\lambda$ ) describing the system in the Clebsch representation, are then the eqs. (2.1), (2.2) and (2.13)–(2.16).

It should be noted that for stationary states (for which the physical fields  $\rho$ ,  $v$  and  $s$  are independent of time) the fields  $\phi_\rho$ ,  $\phi_\lambda$  and  $\phi_s$  are not necessarily independent of time<sup>4</sup>). This is obvious for the case of a homogeneous fluid at rest for which it follows from eq. (2.16), that  $\phi_s = Tt$ . Alternatively, for steady flow Bernoulli's theorem states that  $\frac{1}{2}v^2 + h$  is constant along a streamline, so that, according to eq. (2.13),

$$\dot{\phi}_\rho + \lambda \dot{\phi}_\lambda + s \dot{\phi}_s = \text{constant}$$

along a streamline. As this constant is in general different along different streamlines, it can not be set equal to zero.

In the next section we will review the derivation of the hydrodynamic equations from a variational principle using the Clebsch representation. We shall then also discuss the Hamiltonian formulation of the hydrodynamic equations.

### 3. Variational principle for ideal fluids and Hamiltonian formulation

Following Seliger and Whitham<sup>4</sup>), the six eqs. (2.1), (2.3) and (2.13)–(2.16) can be found as Euler–Lagrange equations from the variational principle

$$\delta \int_{t_1}^{t_2} dt \int dr \mathcal{L} = 0, \quad (3.1)$$

where the Lagrangian density  $\mathcal{L}$  is defined as

$$\mathcal{L} = \rho \left\{ \dot{\phi}_\rho + \lambda \dot{\phi}_\lambda + s \dot{\phi}_s - \frac{1}{2} (\nabla \phi_\rho + \lambda \nabla \phi_\lambda + s \nabla \phi_s)^2 - u(\rho, s) \right\}. \quad (3.2)$$

Here  $u$  is the internal energy per unit of mass, taken as a function of  $\rho$  and  $s$ , for which one has the thermodynamic relations

$$\left( \frac{\partial u}{\partial \rho} \right)_s = \frac{p}{\rho^2}, \quad \left( \frac{\partial u}{\partial s} \right)_\rho = T. \quad (3.3)$$

Eq. (3.1) must hold for arbitrary variations of the fields which vanish at  $t_1$  and  $t_2$  and at the boundary of the volume  $V$  occupied by the fluid. Thus one finds for arbitrary variations of the density  $\rho$

$$\mathcal{L} = p. \tag{3.4}$$

This equation expresses the fact that the Lagrangian density reduces to the pressure on the extremal path and is equivalent to eq. (2.13) since  $h = u + p/\rho$ . For arbitrary variations with respect to  $\phi_\rho$ ,  $\phi_s$ ,  $\phi_\lambda$ ,  $\lambda$  and  $s$  one finds in the same way eqs. (2.1), (2.3) and (2.14)–(2.16) respectively as Euler–Lagrange equations. As discussed in the previous section, these equations are, together with eq. (3.4), the hydrodynamic equations in the Clebsch representation\*.

Clebsch, in his masterful 1859 paper<sup>1)</sup>, showed already that a variational principle could be formulated for the case that the equation of motion for the velocity field of the fluid can be written in the form

$$\frac{dv}{dt} = -\nabla\psi \tag{3.5}$$

if one uses the representation

$$v = -\nabla\phi_\rho - \lambda\nabla\phi_\lambda. \tag{3.6}$$

Eq. (3.5) holds for a number of cases in which the hydrodynamic equations are constrained. Examples are: 1. the case of incompressible flow which was considered by Clebsch himself and for which  $\psi = p/\rho$  (cf. eq. (2.5)); 2. the case of isentropic flow, for which  $\psi = h$  (cf. eq. (2.12)); and 3. isothermal flow, for which  $\psi = h - Ts$ . The work of Bateman<sup>3)</sup> comprises both case 2 and 3. If the equations are not constrained however, the equation of motion is not of the form (3.5). It is then convenient to use the more general Clebsch representation, eq. (2.8).

From the Lagrangian formulation of the hydrodynamic equations one may go over to a Hamiltonian formalism. Contrary to the usual case however, we will obtain the same number of Hamiltonian equations, since the Lagrangian equations are already differential equations of first order in time. As the time derivatives of  $\rho$ ,  $\lambda$  and  $s$  do not appear in the Lagrangian, one can only define the three momenta conjugate to the fields  $\phi_\rho$ ,  $\phi_\lambda$  and  $\phi_s$ ,

$$\Pi_\rho \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_\rho} = \rho, \tag{3.7}$$

\* The derivation of the energy and momentum conservation laws on the basis of Noether's theorem is straightforward and in fact completely parallels the corresponding analysis by Thellung<sup>2)</sup> for isentropic flow, to which the reader is referred.

$$\Pi_\lambda \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\lambda} = \rho \lambda, \quad (3.8)$$

$$\Pi_s \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_s} = \rho s \equiv s_v. \quad (3.9)$$

It is seen that these momenta are the fields  $\rho$ ,  $\rho \lambda$  and  $s_v$ , the entropy per unit of volume. The Hamiltonian density, given by

$$\mathcal{H} \equiv \dot{\phi}_\rho \Pi_\rho + \dot{\phi}_\lambda \Pi_\lambda + \dot{\phi}_s \Pi_s - \mathcal{L}, \quad (3.10)$$

becomes

$$\mathcal{H} = \frac{1}{2}(\Pi_\rho \nabla \phi_\rho + \Pi_\lambda \nabla \phi_\lambda + \Pi_s \nabla \phi_s)^2 / \Pi_\rho + \Pi_\rho u(\Pi_\rho, \Pi_s). \quad (3.11)$$

The hydrodynamic equations may now be found as Hamiltonian equations of motion:

$$\dot{\phi}_\rho = \frac{\partial \mathcal{H}}{\partial \Pi_\rho} = -\mathbf{v} \cdot \nabla \phi_\rho - \frac{1}{2} v^2 + \mu, \quad (3.12)$$

$$\dot{\phi}_\lambda = \frac{\partial \mathcal{H}}{\partial \Pi_\lambda} = -\mathbf{v} \cdot \nabla \phi_\lambda, \quad (3.13)$$

$$\dot{\phi}_s = \frac{\partial \mathcal{H}}{\partial \Pi_s} = -\mathbf{v} \cdot \nabla \phi_s + T, \quad (3.14)$$

$$\dot{\Pi}_\rho = \nabla \cdot \frac{\partial \mathcal{H}}{\partial \nabla \phi_\rho} = -\nabla \cdot \Pi_\rho \mathbf{v}, \quad (3.15)$$

$$\dot{\Pi}_\lambda = \nabla \cdot \frac{\partial \mathcal{H}}{\partial \nabla \phi_\lambda} = -\nabla \cdot \Pi_\lambda \mathbf{v}, \quad (3.16)$$

$$\dot{\Pi}_s = \nabla \cdot \frac{\partial \mathcal{H}}{\partial \nabla \phi_s} = -\nabla \cdot \Pi_s \mathbf{v}, \quad (3.17)$$

where the chemical potential  $\mu$  per unit of mass is given by

$$\mu \equiv u - Ts + p/\rho = \partial(\Pi_\rho u(\Pi_\rho, \Pi_s))/\partial \Pi_\rho. \quad (3.18)$$

Eqs. (3.13)–(3.17) are identical with eqs. (2.1), (2.3) and (2.14)–(2.16). Eq. (3.12) can be shown to be identical to eq. (2.13) by using once more eqs. (3.13) and (3.14).

As one would expect, the Hamiltonian density eq. (3.11) represents the total energy density  $e_v$ :

$$\mathcal{H} = e_v \equiv \frac{1}{2} \mathbf{j}^2 / \rho + \rho u, \quad (3.19)$$

with  $\mathbf{j}$  the momentum density given by, cf. eqs. (2.8) and (3.7)–(3.9),

$$\mathbf{j} \equiv \rho \mathbf{v} = -\Pi_\rho \nabla \phi_\rho - \Pi_\lambda \nabla \phi_\lambda - \Pi_s \nabla \phi_s. \quad (3.20)$$

By introducing the total Hamiltonian

$$H \equiv \int dr \mathcal{H}(r), \quad (3.21)$$

the Hamiltonian equations of motion (3.12)–(3.17) can also be written as

$$\dot{\phi}_k = \frac{\delta H}{\delta \Pi_k}, \quad \dot{\Pi}_k = -\frac{\delta H}{\delta \phi_k}, \quad \text{with } k = \rho, \lambda, s, \quad (3.22)$$

where the functional derivatives  $\delta/\delta\phi_k$  and  $\delta/\delta\Pi_k$  are defined in the standard way<sup>9</sup>).

We now define Poisson-brackets

$$\{A, B\} \equiv \int dr \sum_{k=\rho,\lambda,s} \left[ \frac{\delta A}{\delta \phi_k(r)} \frac{\delta B}{\delta \Pi_k(r)} - \frac{\delta A}{\delta \Pi_k(r)} \frac{\delta B}{\delta \phi_k(r)} \right], \quad (3.23)$$

where  $A$  and  $B$  are arbitrary functionals of the canonical fields. The corresponding Poisson-bracket for either isothermal or isentropic flow was introduced recently by Enz and Turski<sup>7</sup>). If we then consider the five physical fields  $\rho$ ,  $j$  and  $e_v$  as components  $\alpha_i$  of a five-dimensional vector field  $\underline{\alpha}$  such that

$$\begin{aligned} \alpha_0 &\equiv \rho, \\ \alpha_i &\equiv j_i, \quad \text{for } i = 1, 2, 3, \\ \alpha_4 &\equiv e_v, \end{aligned} \quad (3.24)$$

we can define the following basic Poisson-brackets:

$$L_{ij}(r, r') \equiv \{\alpha_i(r), \alpha_j(r')\} = -L_{ji}(r', r), \quad \text{for } i, j = 0, \dots, 4. \quad (3.25)$$

These matrix elements can be evaluated in a straightforward way (see appendix); one finds

$$\begin{aligned} L_{00}(r, r') &= 0, \\ L_{0i}(r, r') &= -L_{i0}(r', r) = -\frac{\partial}{\partial r'_i} [\rho(r)\delta(r-r')], \quad \text{for } i = 1, 2, 3, \\ L_{04}(r, r') &= -L_{40}(r', r) = -\nabla \cdot [j(r)\delta(r-r')], \\ L_{ij}(r, r') &= -L_{ji}(r, r') = -\frac{\partial}{\partial r'_j} [j_i(r)\delta(r-r')] \\ &\quad + \frac{\partial}{\partial r'_i} [j_j(r')\delta(r-r')], \quad \text{for } i, j = 1, 2, 3, \\ L_{i4}(r, r') &= -L_{4i}(r', r) = -\nabla \cdot [v(r)j_i(r)\delta(r-r')] - \frac{\partial}{\partial r'_i} [p(r)\delta(r-r')] \\ &\quad + \frac{\partial}{\partial r'_i} [e_v(r')\delta(r-r')], \quad \text{for } i = 1, 2, 3, \\ L_{44}(r, r') &= -\nabla \cdot [v(r)(e_v(r) + p(r))\delta(r-r')] \\ &\quad + \nabla' \cdot [v(r')(e_v(r') + p(r'))\delta(r-r')]. \end{aligned} \quad (3.26)$$

It is important to note that  $L$  depends on the six canonical fields only via the five physical fields. It is therefore possible, now that these basic Poisson-brackets have been evaluated within the canonical formalism, to define Poisson-brackets for functionals  $A(\{\underline{\alpha}\})$  and  $B(\{\underline{\alpha}\})$  of the physical fields entirely in terms of the matrix  $L$  without reference to the canonical fields:

$$\{A(\{\underline{\alpha}\}), B(\{\underline{\alpha}\})\} \equiv \sum_{i,j=0}^4 \int d\mathbf{r} d\mathbf{r}' \frac{\delta A}{\delta \alpha_i(\mathbf{r})} L_{ij}(\mathbf{r}, \mathbf{r}') \frac{\delta B}{\delta \alpha_j(\mathbf{r}')}. \quad (3.27)$$

With the help of these Poisson-brackets the hydrodynamic equations for the physical fields can be written in the compact form

$$\begin{aligned} \frac{\partial \alpha_i(\mathbf{r}, t)}{\partial t} &= \{\alpha_i(\mathbf{r}, t), H\} \\ &= \sum_j \int d\mathbf{r}' L_{ij}(\mathbf{r}, \mathbf{r}') \frac{\delta H}{\delta \alpha_j(\mathbf{r}', t)} \\ &= \int d\mathbf{r}' L_{i4}(\mathbf{r}, \mathbf{r}'), \quad \text{for } i = 0, \dots, 4. \end{aligned} \quad (3.28)$$

Here we have used the fact that the Hamiltonian is a functional of  $\alpha_4$  only so that

$$\frac{\delta H}{\delta \alpha_i(\mathbf{r}')} = \delta_{i4}. \quad (3.29)$$

With relations (3.26) and the identification (3.24), eqs. (3.28) reduce to

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}, \quad (3.30)$$

$$\frac{\partial \mathbf{j}}{\partial t} = -\nabla \cdot [\mathbf{v}\mathbf{j} + \mathbf{p}], \quad (3.31)$$

$$\frac{\partial \mathbf{e}_v}{\partial t} = -\nabla \cdot [\mathbf{v}(\mathbf{e}_v + \mathbf{p})]. \quad (3.32)$$

This set of equations is equivalent to the set (2.1)–(2.3). The use of the conserved quantities  $\rho$ ,  $\mathbf{j}$  and  $\mathbf{e}_v$  will turn out to have certain advantages.

As a final remark, we note that for the total momentum

$$\mathbf{P} \equiv \int d\mathbf{r} \mathbf{j}(\mathbf{r}), \quad (3.33)$$

one finds with the aid of eqs. (3.26)

$$\{\mathbf{P}, \alpha_i(\mathbf{r})\} = \nabla \alpha_i(\mathbf{r}), \quad \text{for } i = 0, \dots, 4. \quad (3.34)$$

Thus the total momentum is, as expected, the generator of spatial translations.



Recently Dzyaloshinskii and Volovick<sup>8</sup>) used this property as a starting point to arrive at expressions (3.28) for “hydrodynamic” Poisson-brackets without the use of the Hamiltonian formulation based on the Clebsch representation.

#### 4. The Liouville equation

In a statistical description one considers an ensemble of fluid systems and introduces a density distribution  $P(\{\underline{\alpha}(\mathbf{r})\}, t)$  in the “phase space” of physical fields. Since the hydrodynamic equations are first order in time, this density obeys a continuity or conservation equation which one would be tempted to write in the form

$$\frac{\partial P(\{\underline{\alpha}(\mathbf{r})\}, t)}{\partial t} = - \sum_i \int d\mathbf{r} \frac{\delta}{\delta \alpha_i(\mathbf{r})} \dot{\alpha}_i(\mathbf{r}) P(\{\underline{\alpha}(\mathbf{r})\}, t). \quad (4.1)$$

However, strictly speaking this equation is a meaningless string of symbols, since it is not clear how the functional derivative  $\delta \dot{\alpha}_i(\mathbf{r}) / \delta \alpha_i(\mathbf{r})$  occurring in eq. (4.1) should be interpreted\*.

A way to avoid this difficulty is to discretize the system in coordinate space, so that the phase space becomes of finite dimensionality\*\*. To this end we divide the fluid into small cubic cells of size  $\Delta^3$ . The position of a cell is denoted by  $\mathbf{r} = \mathbf{n}\Delta$  where  $\mathbf{n}$  is a vector of which the components are integer numbers. The state of the fluid in each cell is characterized by the five hydrodynamic variables

$$\underline{\alpha}_n = (\rho_n, \mathbf{j}_n, e_{v,n}), \quad (4.2)$$

the specific quantities in each cell. In the limit of vanishing cell-size these variables correspond to the previously defined hydrodynamic fields. The discrete hydrodynamic variables are postulated to obey equations of motion which are discrete analogues of the conservation laws (3.30)–(3.32), viz.

$$\frac{\partial \rho_n}{\partial t} = - \nabla_n \cdot \mathbf{j}_n, \quad (4.3)$$

$$\frac{\partial \mathbf{j}_n}{\partial t} = - \nabla_n \cdot [\mathbf{v}_n \mathbf{j}_n + \mathbf{p}_n], \quad (4.4)$$

$$\frac{\partial e_{v,n}}{\partial t} = - \nabla_n \cdot [(e_{v,n} + p_n) \mathbf{v}_n], \quad (4.5)$$

\*In the previous sections a similar difficulty did not occur since the functional derivatives occurring there had always a well-defined meaning.

\*\* Discretization is in fact also necessary in order to normalize the distribution function.

where  $v_n \equiv j_n/\rho_n$  is the discrete velocity and  $p_n$  the pressure in the cell labeled by  $n$ . The discrete operator  $\nabla_n$  is defined as

$$\nabla_n A_n \equiv \sum_{i=1}^3 \frac{\hat{e}_i}{2\Delta} [A_{n+\hat{e}_i} - A_{n-\hat{e}_i}], \quad (4.6)$$

with the  $\hat{e}_i$ 's unit vectors along the Cartesian axes. The continuity-equation for the distribution function  $P(\{\alpha_n\}, t)$  in the phase space spanned by the discrete set of variables (4.2) is

$$\frac{\partial P(\{\alpha_n\}, t)}{\partial t} = - \sum_n \sum_{i=0}^4 \frac{\partial}{\partial \alpha_{i,n}} \dot{\alpha}_{i,n} P(\{\alpha_n\}, t). \quad (4.7)$$

It is now straightforward to show with the aid of eqs. (4.3)–(4.5) and the definition (4.6) that the following property holds:

$$\sum_i \frac{\partial \dot{\alpha}_{i,n}}{\partial \alpha_{i,n}} = 0. \quad (4.8)$$

This implies that the flow in *this particular phase space is incompressible*. Consequently eq. (4.7) reduces to

$$\frac{\partial P(\{\alpha_n\}, t)}{\partial t} = - \sum_n \sum_{i=0}^4 \dot{\alpha}_{i,n} \frac{\partial P(\{\alpha_n\}, t)}{\partial \alpha_{i,n}}. \quad (4.9)$$

In the limit of continuous fields this equation becomes

$$\frac{\partial P(\{\alpha(\mathbf{r})\}, t)}{\partial t} = - \sum_{i=0}^4 \int d\mathbf{r} \dot{\alpha}_i(\mathbf{r}) \frac{\delta P(\{\alpha(\mathbf{r})\}, t)}{\delta \alpha_i(\mathbf{r})}, \quad (4.10)$$

provided that the limiting distribution function is a proper functional of  $\alpha(\mathbf{r})^9$ . Using also eqs. (3.27) and (3.28) this equation may also be written in the form

$$\frac{\partial P}{\partial t} = \{H, P\}. \quad (4.11)$$

This equation, in which functional derivatives have a well-defined meaning, is a Liouville-equation for the density distribution in the space of physical fields  $\rho$ ,  $\mathbf{j}$  and  $e_v$ . It can be used as the basis for the discussion of the statistical properties of an ideal fluid.

Clearly one could in principle also have derived a Liouville-equation for a distribution in the phase space of the six canonical fields. It is however the reduced density distribution in the space of five physical fields which is directly relevant for the evaluation of averages of physical quantities.

It should be stressed that while an equation of the form (4.7) is also valid for a transformed set of hydrodynamic variables  $\beta_n(\alpha_n)$ , the property (4.8) in this transformed set does not necessarily hold, nor is there then a Liouville-

equation. Using eq. (4.8), one can indeed show in a straightforward way that

$$\sum_n \sum_i \frac{\partial \dot{\beta}_{i,n}}{\partial \beta_{i,n}} = \sum_n \sum_{i,j,k} \dot{\alpha}_{k,n} \frac{\partial \alpha_{j,n}}{\partial \beta_{i,n}} \frac{\partial^2 \beta_{i,n}}{\partial \alpha_{j,n} \partial \alpha_{k,n}}. \quad (4.12)$$

Using also the identity<sup>10)</sup>

$$\frac{\partial \ln \det \mathbf{G}}{\partial x} = \sum_{ij} G_{ij}^{-1} \frac{\partial G_{ij}}{\partial x}, \quad (4.13)$$

which holds for any matrix  $\mathbf{G}$  which depends on a parameter  $x$ , eq. (4.12) can be re-written as

$$\sum_n \sum_i \frac{\partial \dot{\beta}_{i,n}}{\partial \beta_{i,n}} = \sum_n \sum_i \dot{\alpha}_{i,n} \frac{\partial \ln J_n}{\partial \alpha_{i,n}} = \sum_n \sum_i \dot{\beta}_{i,n} \frac{\partial \ln J_n}{\partial \beta_{i,n}}, \quad (4.14)$$

where  $J_n$  is the Jacobian of the transformation  $\underline{\beta}_n \rightarrow \underline{\alpha}_n$ . Thus the flow in  $\underline{\beta}$ -space will only be incompressible if the Jacobian of the transformation to new hydrodynamic variables is a constant. Let us in particular consider three different sets of variables  $\underline{\beta}$ :

a.  $\underline{\beta}_n = (\rho_n, j_n, u_{v,n})$ .

Here  $u_{v,n} = e_{v,n} - \frac{1}{2} j_n^2 / \rho_n$  is the discrete internal energy per unit of volume. The Jacobian for this case is 1, so that the flow in the corresponding phase space is also incompressible.

b.  $\underline{\beta}_n(\rho_n, v_n, e_{v,n})$ .

Here  $v_n \equiv j_n / \rho_n$  is the discrete velocity. The Jacobian of the transformation  $\underline{\beta}_n \rightarrow \underline{\alpha}_n$  becomes in this case

$$J_n = \rho_n^{-3}, \quad (4.15)$$

and one has according to equation (4.14) the result

$$\sum_n \sum_i \frac{\partial \dot{\beta}_{i,n}}{\partial \beta_{i,n}} = -3 \sum_n \dot{\rho}_n / \rho_n. \quad (4.16)$$

c.  $\underline{\beta}_n = (\rho_n, j_n, s_{v,n})$ .

Here  $s_{v,n}$  is the discrete entropy per unit of volume, which is related to the energy density variable via the Gibbs-relation

$$T_n ds_{v,n} = du_{v,n} - \mu_n d\rho_n. \quad (4.17)$$

$T_n$  and  $\mu_n$  are the temperature and the chemical potential in cell  $n$  respectively. The Jacobian becomes in this case

$$J_n = T_n^{-1}. \quad (4.18)$$

Hence one obtains here

$$\sum_n \sum_i \frac{\partial \beta_{i,n}}{\partial \beta_{i,n}} = - \sum_n \left[ \frac{\dot{s}_{v,n}}{T_n} \left( \frac{\partial T_n}{\partial s_{v,n}} \right)_\rho + \frac{\dot{\rho}_n}{T_n} \left( \frac{\partial T_n}{\partial \rho_n} \right)_{s_v} \right]. \quad (4.19)$$

Both in case 2 and 3 the flow in “phase space” is *not* incompressible and a Liouville-equation does not hold. Moreover, it is clear from the general form of eq. (4.14) and the results in the special cases considered above, that the “rate of compression” is either zero (e.g. case 1) or infinite (e.g. cases 2 and 3) in the continuum-limit. In the latter cases the continuum-limit of eq. (4.7) does not exist and we can not give a well-defined meaning to the functional derivative occurring in eq. (4.1)\*.

A final remark must be made in connection with case c. The validity of eq. (4.8) for the original set of variables (4.2) is an immediate consequence of the fact that these variables obey the conservation laws (4.3)–(4.5). On the basis of these equations, using also eq. (4.17), the discrete entropy  $s_{v,n}$  then satisfies an evolution-equation\*\*

$$\begin{aligned} \frac{\partial s_{v,n}}{\partial t} = & \frac{1}{T_n} \left[ - \nabla_n \cdot \left\{ \left( \frac{1}{2} \rho_n v_n^2 + u_{v,n} + p_n \right) \mathbf{v}_n \right\} - \frac{1}{2} v_n^2 (\nabla_n \cdot \mathbf{j}_n) \right. \\ & \left. + \{ \nabla_n \cdot (\mathbf{v}_n \mathbf{j}_n + p_n) \} \cdot \mathbf{v}_n + \mu_n (\nabla_n \cdot \mathbf{j}_n) \right]. \end{aligned} \quad (4.20)$$

Thus the discrete entropy is not conserved (even though eq. (4.20) reduces to eq. (2.3) in the continuum-limit). In this connection it should be remembered that discontinuities such as shock waves in an ideal fluid give rise to an increase of entropy<sup>11)</sup>. On the other hand, if one had started by postulating that  $\rho_n$ ,  $\mathbf{j}_n$  and  $s_{v,n}$  obey the discrete analogues of eqs. (2.1)–(2.3), the discrete energy  $e_{v,n}$  would not have been conserved. As only a description in which mass, momentum and energy are conserved seems satisfying, we have based our discussion on eqs. (4.3)–(4.5).

## 5. The equilibrium distribution

According to the Liouville-equation (4.11), the distribution function  $P(\{\underline{\alpha}(r)\}, t)$  in the phase space of the basic physical fields (3.24) is stationary

\* An alternative way to give meaning to eq. (4.1) would have been to consider the phase space spanned by the Fourier-components of the particular set of hydrodynamic variables (3.24) and to introduce an ad hoc cut-off wave-vector for these variables. If such a procedure is carried out consistently, one obtains in a less transparent way results equivalent to those found above.

\*\* One may easily verify that the result eq. (4.19) can also be obtained from eq. (4.20), together with eqs. (4.3) and (4.4).

if  $P$  is only a function of the constants of the motion of the system, such as its total energy  $E$ , its total entropy  $S$  and its total mass  $M$

$$E = \int dr e_v(\mathbf{r}), \quad S = \int dr s_v(\mathbf{r}) \quad \text{and} \quad M = \int dr \rho(\mathbf{r}). \quad (5.1)$$

For a system which is energetically and materially insulated, the equilibrium distribution is given by the Einstein formula

$$P^{\text{eq}}(\{\alpha(\mathbf{r})\}) \sim e^{S(\{\alpha(\mathbf{r})\})/k}, \quad (5.2)$$

where  $k$  is Boltzmann's constant. This distribution can also be written in the form

$$P^{\text{eq}}(\{\alpha(\mathbf{r})\}) = P^{\text{eq}}(\{\alpha^0(\mathbf{r})\}) e^{\Delta S(\{\alpha(\mathbf{r})\})}, \quad (5.3)$$

where

$$\Delta S(\{\alpha(\mathbf{r})\}) \equiv S(\{\alpha(\mathbf{r})\}) - S^0(\{\alpha^0(\mathbf{r})\}) = \int dr (s_v(\alpha(\mathbf{r})) - s_v(\alpha^0(\mathbf{r}))). \quad (5.4)$$

Here the superscript zero refers to the "equilibrium state" characterised by a uniform density and energy-field and a vanishing momentum density field.

If the system is in thermal contact with a heat-bath with temperature  $T_0$  the equilibrium-distribution is<sup>12)</sup>

$$P^{\text{eq}}(\{\alpha(\mathbf{r})\}) = P^{\text{eq}}(\{\alpha^0(\mathbf{r})\}) e^{-(\Delta E - T_0 \Delta S)/kT_0}, \quad (5.5)$$

while for a system which can in addition exchange mass with a reservoir, this distribution is

$$P^{\text{eq}}(\{\alpha(\mathbf{r})\}) = P^{\text{eq}}(\{\alpha^0(\mathbf{r})\}) e^{-(\Delta E - T_0 \Delta S - \mu_0 \Delta M)/kT_0}, \quad (5.6)$$

where  $\mu_0$  is the thermal chemical potential of the reservoir.

The distributions (5.5) and (5.6) follow immediately from eq. (5.3). One has indeed according to this equation for the system including the bath

$$P^{\text{eq}} \sim e^{(\Delta S + \Delta S_{\text{bath}})/k}. \quad (5.7)$$

For the bath we may write in good approximation

$$\Delta S_{\text{bath}} = \frac{1}{T_0} \Delta E_{\text{bath}} - \frac{\mu_0}{T_0} \Delta M_{\text{bath}}. \quad (5.8)$$

Since the total energy and mass of the system together with the bath are constant,  $\Delta E_{\text{bath}} = -\Delta E$  and  $\Delta M_{\text{bath}} = -\Delta M$ , the distribution (5.7) reduces to the distribution (5.6). In the same way eq. (5.5) is found if the system is materially insulated, i.e. if  $\Delta M = 0$ .

It should be stressed that the distribution functions (5.3), (5.5) and (5.6) are

equilibrium distribution functions in the phase space of the fields  $\rho$ ,  $j$  and  $e_v$ . If one had considered a different phase space, e.g. the one spanned by the fields  $\rho$ ,  $v$  and  $e_v$ , the above distributions must be divided by the Jacobian of the corresponding transformation\*, which in general does not exist in the continuum-limit. For linearized hydrodynamics, this problem does not arise as the transformations considered are then always linear so that the Jacobian is a constant.

## 6. Discussion

In the preceding sections we have used the Hamiltonian formalism based on the canonical fields of the extended Clebsch representation introduced by Seliger and Whitham<sup>3)</sup> to obtain expressions for hydrodynamic Poisson-brackets. With the help of these, we were able to formulate a Liouville-equation in the phase space of the particular set of (non-canonical) physical fields  $\rho$ ,  $j$  and  $e_v$ . The well-known equilibrium distribution functions for the fluid fluctuations are stationary solutions of this Liouville-equation. This fact plays an important role in an analysis of nonlinear hydrodynamic fluctuations as we shall discuss in a subsequent paper.

## Appendix

In this appendix we evaluate the Poisson-brackets defined by eq. (3.25) for the physical fields  $\rho$ ,  $j$  and  $e_v$ , which are given in terms of the canonical fields by (cf. eqs. (3.7)–(3.9), (3.19) and (3.20))

$$\rho = \Pi_\rho, \quad (\text{A.1})$$

$$j = -\Pi_\rho \nabla \phi_\rho - \Pi_\lambda \nabla \phi_\lambda - \Pi_s \nabla \phi_s, \quad (\text{A.2})$$

$$e_v = \frac{1}{2}(\Pi_\rho \nabla \phi_\rho + \Pi_\lambda \nabla \phi_\lambda + \Pi_s \nabla \phi_s)^2 / \Pi_\rho + u_v(\Pi_\rho, \Pi_s). \quad (\text{A.3})$$

Here  $u_v = \rho u$  is the internal energy per unit of volume. We will first evaluate the Poisson-brackets for  $\rho$ ,  $j$  and  $u_v$ .

\* Einstein<sup>13)</sup> in his original article on fluctuation theory includes in the distribution function in addition to the exponential factor  $\exp(\Delta S/k)$  a function  $f$  which depends on the choice of the fluctuating variables and which plays the same role as the Jacobian in the above discussion. He then argues that if  $\Delta S$  may be approximated by a quadratic function, the function  $f$  may be replaced by a constant  $f$ . This corresponds to the linear theory.

Since the Poisson-brackets of momenta vanish, we immediately obtain

$$L_{00} = \{\rho, \rho\} = 0, \quad (\text{A.4})$$

$$\{\rho, u_v\} = 0, \quad (\text{A.5})$$

$$\{u_v, u_v\} = 0. \quad (\text{A.6})$$

From the definition (A.2) we get

$$\frac{\delta \mathbf{j}(\mathbf{r})}{\delta \phi_\alpha(\mathbf{r}')} = \nabla'(\Pi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')), \quad (\alpha = \rho, \lambda, s) \quad (\text{A.7})$$

$$\frac{\delta \mathbf{j}(\mathbf{r})}{\delta \Pi_\alpha(\mathbf{r})} = -\nabla \phi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}'), \quad (\alpha = \rho, \lambda, s) \quad (\text{A.8})$$

where  $\nabla' = \partial/\partial \mathbf{r}'$ . One therefore obtains

$$\begin{aligned} \{\Pi_\alpha(\mathbf{r}), j_i(\mathbf{r}')\} &= - \int d\mathbf{r}'' \frac{\delta \Pi_\alpha(\mathbf{r})}{\delta \Pi_\alpha(\mathbf{r}'')} \frac{\delta j_i(\mathbf{r}')}{\delta \phi_\alpha(\mathbf{r}'')} \\ &= - \int d\mathbf{r}'' \delta(\mathbf{r} - \mathbf{r}'') \nabla''_i(\Pi_\alpha(\mathbf{r}'')\delta(\mathbf{r} - \mathbf{r}'')) \\ &= -\nabla_i(\Pi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')), \quad (i = 1, 2, 3; \alpha = \rho, \lambda, s). \end{aligned} \quad (\text{A.9})$$

Thus we have

$$L_{0i}(\mathbf{r}, \mathbf{r}') = \{\rho(\mathbf{r}), j_i(\mathbf{r}')\} = -\nabla_i(\rho(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')) \quad (i = 1, 2, 3). \quad (\text{A.10})$$

Similarly we obtain

$$\begin{aligned} L_{ik}(\mathbf{r}, \mathbf{r}') = \{j_i(\mathbf{r}), j_k(\mathbf{r}')\} &= - \int d\mathbf{r}'' \sum_{\alpha=\rho,\lambda,s} [\nabla''_i(\Pi_\alpha(\mathbf{r}'')\delta(\mathbf{r} - \mathbf{r}'')) \\ &\quad \times \nabla'_k \phi_\alpha(\mathbf{r}')\delta(\mathbf{r}' - \mathbf{r}'') \\ &\quad - \nabla_i \phi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'') \nabla''_k(\Pi_\alpha(\mathbf{r}'')\delta(\mathbf{r}' - \mathbf{r}''))] \\ &= - \sum_{\alpha=\rho,\lambda,s} [\nabla'_k \phi_\alpha(\mathbf{r}') \nabla'_i(\Pi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) \\ &\quad - \nabla_i \phi_\alpha(\mathbf{r}) \nabla'_k(\Pi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'))] \\ &= - \sum_{\alpha=\rho,\lambda,s} [\nabla'_i(\Pi_\alpha(\mathbf{r}') \nabla'_k \phi_\alpha(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) \\ &\quad - \nabla_k(\Pi_\alpha(\mathbf{r}) \nabla'_i \phi_\alpha(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}'))] \\ &= \nabla'_i(j_k(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')) \\ &\quad - \nabla_k(j_i(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')) \quad (i, k = 1, 2, 3). \end{aligned} \quad (\text{A.11})$$

From the thermodynamic relation

$$du_v = T ds_v + \mu d\rho, \quad (\text{A.12})$$

one finds that the derivatives of  $u_v$  with respect to  $s_v (= \rho s = \Pi_s)$  and  $\rho (= \Pi_\rho)$

are  $T$  and  $\mu$  respectively. With eq. (A.9) and the definition (3.13), one then gets

$$\begin{aligned} \{u_v(\mathbf{r}), j_i(\mathbf{r}')\} &= -\mu(\mathbf{r})\nabla_i(\rho(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) - T(\mathbf{r})\nabla_i(s_v(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) \\ &= -(u_v(\mathbf{r}) + p(\mathbf{r}))\nabla_i\delta(\mathbf{r}-\mathbf{r}') \\ &\quad - (\mu(\mathbf{r})\nabla_i\rho(\mathbf{r}) + T(\mathbf{r})\nabla_i s_v(\mathbf{r}))\delta(\mathbf{r}-\mathbf{r}') \quad (i = 1, 2, 3). \end{aligned} \quad (\text{A.13})$$

Using eq. (A.12) once more, we can rewrite eq. (A.13) as

$$\{u_v(\mathbf{r}), j_i(\mathbf{r}')\} = -\nabla_i(u_v(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')) - p(\mathbf{r})\nabla_i\delta(\mathbf{r}-\mathbf{r}') \quad (i = 1, 2, 3). \quad (\text{A.14})$$

With the aid of the Poisson-brackets (A.4), (A.5), (A.11) and (A.14) we now find

$$\begin{aligned} L_{04}(\mathbf{r}, \mathbf{r}') &= \{\rho(\mathbf{r}), e_v(\mathbf{r}')\} = \{\rho(\mathbf{r}), j^2(\mathbf{r}')/2\rho(\mathbf{r}') + u_v(\mathbf{r}')\} \\ &= \{\rho(\mathbf{r}), \mathbf{j}(\mathbf{r}')\} \cdot \mathbf{v}(\mathbf{r}') = -\nabla \cdot (\mathbf{j}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')), \end{aligned} \quad (\text{A.15})$$

and

$$\begin{aligned} L_{i4}(\mathbf{r}, \mathbf{r}') &= \{j_i(\mathbf{r}), e_v(\mathbf{r}')\} = \{j_i(\mathbf{r}), j^2(\mathbf{r}')/2\rho(\mathbf{r}') + u_v(\mathbf{r}')\}, \\ &= \{j_i(\mathbf{r}), \mathbf{j}(\mathbf{r}')\} \cdot \mathbf{v}(\mathbf{r}') - \{j_i(\mathbf{r}), \rho(\mathbf{r}')\} \frac{1}{2}v^2(\mathbf{r}') + \{j_i(\mathbf{r}), u_v(\mathbf{r}')\}, \\ &= \nabla'_i(\mathbf{j}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) \cdot \mathbf{v}(\mathbf{r}') - \nabla'_i(j_i(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) \cdot \mathbf{v}(\mathbf{r}') \\ &\quad - \frac{1}{2}v^2(\mathbf{r}')\nabla'_i(\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) + \nabla'_i(u_v(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) + p(\mathbf{r}')\nabla'_i\delta(\mathbf{r}-\mathbf{r}'), \\ &= -\nabla \cdot (\mathbf{j}(\mathbf{r}')v_i(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) - \nabla'_i(p(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) \\ &\quad + \nabla'_i(e_v(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')) \quad (i = 1, 2, 3). \end{aligned} \quad (\text{A.16})$$

The Poisson-bracket  $\{e_v, e_v\}$  can be written as

$$\begin{aligned} L_{44}(\mathbf{r}, \mathbf{r}') &= \{e_v(\mathbf{r}), e_v(\mathbf{r}')\} = -\frac{1}{2}v^2(\mathbf{r})\{\rho(\mathbf{r}), e_v(\mathbf{r}')\} + \mathbf{v}(\mathbf{r}) \cdot \{\mathbf{j}(\mathbf{r}), e_v(\mathbf{r}')\} \\ &\quad + \{u_v(\mathbf{r}), \mathbf{j}(\mathbf{r}')\} \cdot \mathbf{v}(\mathbf{r}'). \end{aligned} \quad (\text{A.17})$$

Upon substitution of eqs. (A.14), (A.15) and (A.16) into eq. (A.17) one finally arrives at

$$\begin{aligned} L_{44}(\mathbf{r}, \mathbf{r}') &= \{e_v(\mathbf{r}), e_v(\mathbf{r}')\} = -\nabla \cdot [\mathbf{v}(\mathbf{r})(e_v(\mathbf{r}) + p(\mathbf{r}))\delta(\mathbf{r}-\mathbf{r}')] \\ &\quad + \nabla' \cdot [\mathbf{v}(\mathbf{r}')(e_v(\mathbf{r}') + p(\mathbf{r}'))\delta(\mathbf{r}-\mathbf{r}')]. \end{aligned} \quad (\text{A.18})$$

This completes the derivation of the expressions (3.26).

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