

## NON-LINEAR HYDRODYNAMIC FLUCTUATIONS AROUND EQUILIBRIUM

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We formulate a scheme describing the fluctuations in a system obeying the non-linear hydrodynamic equations. The random fluxes are assumed to be Gaussian processes with white noise. It is shown that the usual expressions for the systematic parts of the dissipative fluxes are consistent with this assumption, provided that the Onsager coefficients are constants. The linear response of the system to a small external force field is studied and the relevant fluctuation-dissipation theorems are derived.

### 1. Introduction

Linear fluctuation theory is well established and has been applied to hydrodynamics with considerable success<sup>1</sup>). Within the framework of such a linear theory, the hydrodynamic equations are *fully linearized*. Hydrodynamics, however, contains non-linear contributions from various sources. On the one hand, there are all those non-linear contributions which are due to the occurrence of reversible convective fluxes and also to the non-linear functional dependence of the thermodynamic functions on the thermodynamic state parameters. On the other hand, even if one wishes to remain within the domain of validity of non-equilibrium thermodynamics, in which case dissipative fluxes are linear functions of the gradients of the hydrodynamic fields, there occur additional non-linearities due to the dependence of the transport coefficients on the state variables, and to the presence of dissipative quadratic source terms such as the Rayleigh dissipation function.

Several methods have been developed to take into account the hydrodynamic non-linearities within the framework of a hydrodynamic fluctuation theory\*. Thus the various approaches to what is generally known as

\* For fluctuations around stationary non-equilibrium states it is customary to first linearise the non-linear equations around the stationary state and then to perform a linear fluctuation analysis<sup>2</sup>). Such a procedure has, for instance, been used to discuss fluctuations near the Bénard instability<sup>3</sup>) and to discuss light-scattering in a stationary temperature gradient. A detailed discussion and an extensive list of references of the latter problem, which has also been studied by other methods<sup>4</sup>), can be found in ref. 5. It should be emphasized that in this paper we consider *non-linear* fluctuations around *equilibrium*.

mode–mode coupling<sup>6</sup>) have led to the introduction of the concept of renormalized transport coefficient, and to the calculation of the long-time behaviour, the long-time tails, exhibited by the hydrodynamic correlation functions<sup>7</sup>). One aspect, however, within the context of the different mode–mode coupling theories of the hydrodynamical equations, is that quite generally not all non-linearities occurring in these equations are retained in the analysis. For instance, the contribution of the Rayleigh dissipation term is usually neglected<sup>8</sup>). It is not clear whether such a procedure is consistent, and, in particular, whether the assumption which is usually made for the random fluxes occurring in the truncated equations, namely that they are Gaussian processes with white noise, is still compatible with the known equilibrium distribution of the hydrodynamic variables.

It is the aim of this paper to formulate a scheme for the description of hydrodynamic fluctuations around equilibrium, in which the non-linearities are retained in a consistent way and in which the assumption is made that the random fluxes are Gaussian processes with white noise. A preliminary version of this work was discussed in ref. 9. In spirit, our discussion is analogous to the one given by Enz and Turski<sup>10</sup>) who, however, restricted their treatment to the case in which temperature fluctuations do not occur, thereby omitting the energy-equation completely.

In a previous paper<sup>11</sup>), hereafter to be referred to as paper I, we developed a Hamiltonian formalism for the hydrodynamics of a one-component fluid. We also discussed the statistical description of such a system and derived the Liouville-equation for the density distribution in the “phase space” of the hydrodynamic fields. These statistical considerations shall now be extended to a real fluid in which one has the dissipative phenomena of heat conduction and viscous pressure.

In section 2 we summarize some of the results of paper I which we use in our subsequent discussion. In section 3 we introduce the stochastic differential equations for the hydrodynamic fields. The random fluxes are assumed to be Gaussian processes with white noise, while the form of the systematic parts of the dissipative currents remains unspecified. On the basis of this assumption we derive in section 4 the corresponding Fokker–Planck equation. This derivation is performed in a well-defined way by a discretization of the hydrodynamic fields. The Fokker–Planck equation which is obtained reduces to the Liouville-equation discussed in paper I in the continuum limit if the dissipative terms are neglected.

The form of the systematic part of the dissipative currents compatible with the known equilibrium distribution and with the assumption of Gaussian white noise for the random currents is derived in section 5. It is found that they have their usual form with the proviso that the Onsager coefficients (i.e. the

viscosities divided by the temperature and the heat conductivity divided by the temperature squared) do not depend on the fluctuating variables.

Finally, in section 6, we study the linear response to a small external force density and establish the various fluctuation–dissipation theorems within the framework of non-linear fluctuating hydrodynamics. The renormalization of transport coefficients may be based on these equations.

## 2. Hamiltonian form of ideal fluid hydrodynamics

In this section we summarize some of the results of paper I, which will be of use in our subsequent discussion of hydrodynamic fluctuations.

We will describe the state of a one-component fluid by the five physical fields  $\rho$ ,  $\mathbf{j}$  and  $e_v$ , the mass-density, the momentum-density and the total energy-density respectively. The internal energy per unit mass  $u$  is related to  $e_v$  by the relation  $e_v = \frac{1}{2}j^2/\rho + \rho u$ . It is convenient to consider these five fields as components  $\alpha_\beta$  of a five-dimensional vector  $\alpha^*$ ,

$$\alpha_0 \equiv \rho, \quad \alpha_i \equiv j_i \text{ for } i = 1, 2, 3, \quad \alpha_4 \equiv e_v. \quad (2.1)$$

As discussed in paper I, Poisson-brackets for these fields may be defined within the framework of a Hamiltonian formalism; these Poisson-brackets have the usual property

$$\{\alpha_i(\mathbf{r}), \alpha_j(\mathbf{r}')\} = -\{\alpha_j(\mathbf{r}'), \alpha_i(\mathbf{r})\}. \quad (2.2)$$

We now introduce a matrix  $\mathbf{L}$  of which the elements coincide with the Poisson-brackets  $\{\alpha_i, \alpha_j\}$  of the five fields (2.1). The elements of this matrix are explicitly

$$\begin{aligned} L_{00}(\mathbf{r}, \mathbf{r}') &\equiv \{\alpha_0(\mathbf{r}), \alpha_0(\mathbf{r}')\} = 0, \\ L_{0i}(\mathbf{r}, \mathbf{r}') &\equiv \{\alpha_0(\mathbf{r}), \alpha_i(\mathbf{r}')\} = -\frac{\partial}{\partial r_i} [\rho(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')], \\ L_{04}(\mathbf{r}, \mathbf{r}') &\equiv \{\alpha_0(\mathbf{r}), \alpha_4(\mathbf{r}')\} = -\nabla \cdot [\mathbf{j}(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')], \\ L_{ij}(\mathbf{r}, \mathbf{r}') &\equiv \{\alpha_i(\mathbf{r}), \alpha_j(\mathbf{r}')\} = -\frac{\partial}{\partial r_j} [j_i(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')] + \frac{\partial}{\partial r'_i} [j_j(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')], \end{aligned} \quad (2.3)$$

\* Throughout this paper, Greek indices ( $\beta, \gamma$ , etc.) run from 0 to 4, whereas Latin indices ( $i, j$ , etc.) can run from 1 to 3 and designate Cartesian components of three-dimensional vectors.

$$L_{i4}(\mathbf{r}, \mathbf{r}') \equiv \{\alpha_i(\mathbf{r}), \alpha_4(\mathbf{r}')\} = -\nabla \cdot [\mathbf{v}(\mathbf{r})j_i(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')] - \frac{\partial}{\partial r_i} [p(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')] \\ + \frac{\partial}{\partial r'_i} [e_v(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')],$$

$$L_{44}(\mathbf{r}, \mathbf{r}') \equiv \{\alpha_4(\mathbf{r}), \alpha_4(\mathbf{r}')\} = -\nabla \cdot [\mathbf{v}(\mathbf{r})(e_v(\mathbf{r}) + p(\mathbf{r}))\delta(\mathbf{r} - \mathbf{r}')] \\ + \nabla' \cdot [\mathbf{v}(\mathbf{r}')(e_v(\mathbf{r}') + p(\mathbf{r}'))\delta(\mathbf{r} - \mathbf{r}')].$$

In these equations  $p$  denotes the hydrostatic pressure and  $\mathbf{v}$  denotes the velocity, defined by  $\mathbf{v} = \mathbf{j}/\rho$ . The other elements of  $\mathbf{L}$  can be obtained from the symmetry-relation (cf. eq. (2.2))

$$L_{ij}(\mathbf{r}, \mathbf{r}') = -L_{ji}(\mathbf{r}', \mathbf{r}). \quad (2.4)$$

Poisson-brackets for arbitrary functionals of the physical fields  $A(\{\alpha\})$  and  $B(\{\alpha\})$  are then given by

$$\{A(\{\alpha\}), B(\{\alpha\})\} = \sum_{\beta, \gamma=1}^4 \int d\mathbf{r} \int d\mathbf{r}' \frac{\delta A}{\delta \alpha_\beta(\mathbf{r})} L_{\beta\gamma}(\mathbf{r}, \mathbf{r}') \frac{\delta B}{\delta \alpha_\gamma(\mathbf{r}')}, \quad (2.5)$$

where the functional derivatives  $\delta/\delta\alpha_\beta(\mathbf{r})$  are defined in the standard way. With these definitions, the hydrodynamic equations of motion for ideal flow may be written in the compact form

$$\frac{\partial \alpha_\beta}{\partial t} = \{\alpha_\beta, H\}. \quad (2.6)$$

The Hamiltonian  $H$  in this equation is equal to the total energy of the fluid,

$$H \equiv \int d\mathbf{r} e_v(\mathbf{r}). \quad (2.7)$$

Since this equation implies that

$$\frac{\delta H}{\delta \alpha_\beta(\mathbf{r})} = \delta_{\beta 4}, \quad (2.8)$$

one finds from eqs. (2.6), (2.5) and (2.3) the equations of motion

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}, \quad (2.9)$$

$$\frac{\partial \mathbf{j}}{\partial t} = -\nabla \cdot \mathbf{v}\mathbf{j} - \nabla p, \quad (2.10)$$

$$\frac{\partial e_v}{\partial t} = -\nabla \cdot [\mathbf{v}(e_v + p)]. \quad (2.11)$$

These are the equations of conservation of mass, momentum and energy for an ideal fluid.

### 3. Stochastic differential equations for hydrodynamic fluctuations

In a non-ideal one component fluid one has the following conservation laws for the mass-density  $\rho(\mathbf{r}, t)$ , the momentum-density  $\mathbf{j}(\mathbf{r}, t)$  and the energy-density  $e_v(\mathbf{r}, t)$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}, \quad (3.1)$$

$$\frac{\partial \mathbf{j}}{\partial t} = -\nabla \cdot [\mathbf{v}\mathbf{j} + \mathbf{p} + \mathbf{\Pi}], \quad (3.2)$$

$$\frac{\partial e_v}{\partial t} = -\nabla \cdot [\mathbf{v}(e_v + p) + \mathbf{v} \cdot \mathbf{\Pi} + \mathbf{J}]. \quad (3.3)$$

Here  $\mathbf{\Pi}(\mathbf{r}, t)$  and  $\mathbf{J}(\mathbf{r}, t)$  are the symmetric viscous pressure tensor and heat current respectively. In macroscopic hydrodynamics the viscous pressure tensor and the heat current obey phenomenological laws which relate these quantities to the hydrodynamic fields and their gradients<sup>12)</sup> so that eqs. (3.1)–(3.3) form a complete set of deterministic equations. In the context of fluctuation theory we shall consider these quantities to consist of systematic parts  $\mathbf{\Pi}^s$  and  $\mathbf{J}^s$ , and random parts  $\mathbf{\Pi}^R$  and  $\mathbf{J}^R$ ,

$$\mathbf{\Pi} = \mathbf{\Pi}^s + \mathbf{\Pi}^R, \quad (3.4)$$

$$\mathbf{J} = \mathbf{J}^s + \mathbf{J}^R. \quad (3.5)$$

The systematic viscous pressure tensor and heat current again obey phenomenological laws but now in terms of the fluctuating hydrodynamic fields. In order that the stochastic differential eqs. (3.1)–(3.3) with (3.4) and (3.5) may be solved, the stochastic properties of the random fluxes must be specified. We shall assume the processes  $\mathbf{\Pi}^R(\mathbf{r}, t)$  and  $\mathbf{J}^R(\mathbf{r}, t)$  to be Gaussian with zero mean,

$$\begin{aligned} \overline{\mathbf{\Pi}^R(\mathbf{r}, t)} &= 0, \\ \overline{\mathbf{J}^R(\mathbf{r}, t)} &= 0, \end{aligned} \quad (3.6)$$

and variances

$$\begin{aligned} \overline{\mathbf{\Pi}_{ij}^R(\mathbf{r}, t)\mathbf{\Pi}_{kl}^R(\mathbf{r}', t')} &= 2kL_{ijkl}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t') \\ \equiv 2k \left[ L \left( \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl} \right) + L_v\delta_{ij}\delta_{kl} \right] \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'), \end{aligned} \quad (3.7)$$

$$\overline{\mathbf{\Pi}_{ij}^R(\mathbf{r}, t)\mathbf{J}_k^R(\mathbf{r}', t')} = 0, \quad (3.8)$$

$$\overline{\mathbf{J}_i^R(\mathbf{r}, t)\mathbf{J}_j^R(\mathbf{r}', t')} = 2kL_q\delta_{ij}\delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \quad (3.9)$$

In these equations averages denoted by a bar are taken over an ensemble of systems with the same initial conditions for the hydrodynamic fields. Eqs. (3.7) and (3.9) express the fact that the correlations of the random fluxes are local in time (white noise) and space. Due to isotropy, the strength of these correlations (which, for convenience, is expressed as a multiple of  $2k$ , where  $k$  is Boltzmann's constant) is completely determined by three scalar quantities  $L$ ,  $L_v$  and  $L_q$ , which may still depend on the equilibrium temperature  $T_0$  and the equilibrium density  $\rho_0$  of the system.

Equations of the form (3.1)–(3.9) were first proposed by Landau and Lifshitz<sup>13</sup>), in combination with the phenomenological laws

$$\mathbf{II}^s = -2\eta \overline{\nabla \mathbf{v}} - \eta_v \nabla \cdot \mathbf{v}, \quad (3.10)$$

$$\mathbf{J}^s = -\lambda \nabla T, \quad (3.11)$$

where  $\eta$ ,  $\eta_v$  and  $\lambda$  are the viscosity, the volume viscosity and the heat conductivity respectively and where  $\overline{\cdot \cdot}$  denotes the symmetric traceless part of a tensor. It is in their completely linearized version that eqs. (3.1)–(3.3) with eqs. (3.4)–(3.11) are usually referred to as the Landau–Lifshitz hydrodynamic equations. In this case it turns out that they form a consistent set of equations in the sense that all moments of the hydrodynamic fields, calculated in the fully linearized scheme, approach for long times their equilibrium values, provided that the following identification is made:

$$L = \eta T_0, \quad L_v = \eta_v T_0, \quad L_q = \lambda T_0^2. \quad (3.12)$$

The relations (3.12) represent the fluctuation–dissipation theorem in the fully linearized case.

It is our aim to investigate under what conditions, or, more precisely, for which form of the phenomenological laws for the systematic currents  $\mathbf{II}^s$  and  $\mathbf{J}^s$ , the *nonlinear* eqs. (3.1)–(3.3) are consistent with the assumptions (3.4)–(3.9), and the assumption of the Gaussian character of the random fluxes. To this end we shall in the next section derive the Fokker–Planck equation which is equivalent to our set of stochastic equations.

#### 4. The Fokker–Planck equation

When considering the stochastic differential eqs. (3.1)–(3.3) together with eqs. (3.4)–(3.9), the problem arises whether these should be interpreted in the Stratonovich or Itô sense<sup>14</sup>), due to the presence of the term  $\mathbf{v} \cdot \mathbf{II}^R$  in eq. (3.3). It is well known that it is possible, if these equations are interpreted in the Stratonovich sense, to derive the corresponding Itô equations and vice

versa<sup>14</sup>). In appendix A we show, using the discretization rules introduced in paper I, that the Itô and Stratonovich equations are in this case identical, so that either of the interpretations may be used. We shall adopt here the Stratonovich interpretation, so that the usual rules of differential and integral calculus may be used.

We now turn to the derivation of the Fokker–Planck equation. In view of the difficulties encountered when writing down a conservation equation for the density distribution in the phase-space of physical fields (see paper I, where this is done in order to obtain the Liouville equation), it is necessary for such a derivation to discretize the system in coordinate space so that the phase-space becomes of finite dimensionality. As in paper I, we divide the fluid into small cubic cells of size  $\Delta^3$ . We denote the position of a cell by  $\mathbf{r} = \mathbf{n}\Delta$ , where  $\mathbf{n}$  is a vector of which the components are integer numbers. In each cell the state is characterized by the five hydrodynamic variables  $\rho_n$ ,  $\mathbf{j}_n$  and  $e_{v,n}$ , the specific quantities in each cell, which in the limit of vanishing cell-size correspond to the previously defined hydrodynamic fields. These discrete hydrodynamic variables, will obey the discrete analogues of the stochastic hydrodynamic eqs. (2.1)–(2.5), viz.

$$\frac{\partial \rho_n}{\partial t} = -\nabla_n \cdot \mathbf{j}_n, \quad (4.1)$$

$$\frac{\partial \mathbf{j}_n}{\partial t} = -\nabla_n \cdot (\mathbf{v}_n \mathbf{j}_n + p_n + \mathbf{\Pi}_n), \quad (4.2)$$

$$\frac{\partial e_{v,n}}{\partial t} = -\nabla_n \cdot [\mathbf{v}_n (e_{v,n} + p_n) + \mathbf{v}_n \cdot \mathbf{\Pi}_n + \mathbf{J}_n], \quad (4.3)$$

$$\mathbf{\Pi}_n = \mathbf{\Pi}_n^s + \mathbf{\Pi}_n^R, \quad (4.4)$$

$$\mathbf{J}_n = \mathbf{J}_n^s + \mathbf{J}_n^R. \quad (4.5)$$

Here  $\mathbf{\Pi}_n$  and  $\mathbf{J}_n$  are the discrete viscous pressure tensor and heat current respectively, and the discrete gradient operator  $\nabla_n$  is defined as (cf. paper I, eq. (4.6))

$$\nabla_n A_n \equiv \sum_{i=1}^3 \frac{\hat{e}_i}{2\Delta} [A_{n+\hat{e}_i} - A_{n-\hat{e}_i}], \quad (4.6)$$

where the  $\hat{e}_i$ 's are unit vectors along the Cartesian axes.

In agreement with our previous assumption for the continuous stochastic fields  $\mathbf{\Pi}^R(\mathbf{r}, t)$  and  $\mathbf{J}^R(\mathbf{r}, t)$ , we assume that also the processes  $\mathbf{\Pi}_n^R(t)$  and  $\mathbf{J}_n^R(t)$  are Gaussian with zero mean,

$$\overline{\mathbf{\Pi}_n^R(t)} = 0, \quad \overline{\mathbf{J}_n^R(t)} = 0, \quad (4.7)$$

and have as variances the discrete analogues of eqs. (3.7)–(3.9),

$$\overline{\Pi_{ij,n}^R(t)\Pi_{kl,n}^R(t')} = 2kL_{ijkl}\Delta^{-3}\delta_{nn'}\delta(t-t'), \quad (4.8)$$

$$\overline{\Pi_{ij,n}^R(t)J_n^R(t')} = 0, \quad (4.9)$$

$$\overline{J_{i,n}^R(t)J_{j,n}^R(t')} = 2kL_q\delta_{ij}\Delta^{-3}\delta_{nn'}\delta(t-t'). \quad (4.10)$$

With the aid of the notation (2.1) the five eqs. (4.1)–(4.3) together with eqs. (4.4) and (4.5) can be written in the compact form

$$\frac{\partial\alpha_{\beta,n}}{\partial t} = F_{\beta,n}^{\text{irr}} + F_{\beta,n}^{\text{rev}} + \sum_{n'} \sum_{\gamma,\delta=0}^4 M_{\beta\gamma\delta,nn'} f_{\gamma\delta,n'}, \quad (\beta = 0, 1, 2, 3, 4). \quad (4.11)$$

In eq. (4.11) the quantities  $F_{\beta,n}^{\text{rev}}$  and  $F_{\beta,n}^{\text{irr}}$  denote the reversible and irreversible parts of the rates of change of the quantities  $\alpha_{\beta,n}$  respectively:

$$F_{0,n}^{\text{rev}} \equiv -\nabla_n \cdot \mathbf{j}_n, \quad F_{i,n}^{\text{rev}} \equiv -(\nabla_n \cdot \mathbf{v}_n \mathbf{j}_n + \nabla_n p_n)_i, \quad F_{4,n}^{\text{rev}} \equiv -\nabla_n \cdot \{(e_{i,n} + p_n)\mathbf{v}_n\}, \quad (4.12)$$

$$F_{0,n}^{\text{irr}} \equiv 0, \quad F_{i,n}^{\text{irr}} \equiv -(\nabla_n \cdot \Pi_n^s)_i, \quad F_{4,n}^{\text{irr}} \equiv -\nabla_n \cdot \{\Pi_n^s \cdot \mathbf{v}_n + \mathbf{J}_n^s\}, \quad (4.13)$$

while the quantities  $M_{\beta\gamma\delta,nn'}$  and the random forces  $f_{\beta\gamma,n}$  may be identified as

$$M_{0\beta\gamma,nn'} = M_{\beta 0\gamma,nn'} = M_{\beta\gamma 0,nn'} = M_{i4\beta,nn'} = M_{\beta\gamma 4,nn'} = 0,$$

$$M_{ijk,nn'} = -\delta_{ik}\nabla_{j,n}\delta_{nn'},$$

$$M_{4jk,nn'} = -v_{k,n}\nabla_{j,n}\delta_{nn'}, \quad (4.14)$$

$$M_{44i,nn'} = -\nabla_{i,n}\delta_{nn'}$$

$$f_{0\beta,n} = f_{\beta 0,n} = f_{44,n} = f_{i4,n} = 0,$$

$$f_{ij,n} = \Pi_{ij,n}^R, \quad (4.15)$$

$$F_{4i,n} = J_{i,n}^R.$$

The properties of the random forces  $f_{\beta\gamma,n}$  follow from the identification (4.15). They are therefore Gaussian and have variances which follow from eqs. (4.7)–(4.9),

$$\overline{f_{\beta\gamma,n}(t)f_{\delta\epsilon,n}(t')} = 2\Lambda_{\beta\gamma\delta\epsilon,nn'}\delta(t-t'), \quad (4.16)$$

where

$$\Lambda_{ijkl,nn'} = kL_{ijkl}\Delta^{-3}\delta_{nn'}, \quad (4.17)$$

$$\Lambda_{4i4j,nn'} = kL_q\delta_{ij}\Delta^{-3}\delta_{nn'}. \quad (4.18)$$

The other components of  $\Lambda$  are all zero.



For the distribution function  $P(\{\alpha_n\}, t)$  one may now derive the Fokker-Planck equation corresponding to the Stratonovich differential equation (4.11) in a standard way<sup>14</sup>). One obtains the equation

$$\begin{aligned} \frac{\partial P(\{\alpha_n\}, t)}{\partial t} = & \sum_n \sum_\beta \frac{\partial}{\partial \alpha_{\beta,n}} \left[ -F_{\beta,n}^{\text{rev}} - F_{\beta,n}^{\text{irr}} \right. \\ & \left. + \sum_{\substack{n',n'' \\ \gamma,\delta}} \sum_{\substack{\gamma',\delta' \\ \beta'}} M_{\beta\gamma\delta,nn'} \frac{\partial}{\partial \alpha_{\beta',n''}} M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n''n'''} \right] P(\{\alpha_n\}, t). \end{aligned} \tag{4.19}$$

If one takes the dissipative terms, i.e.  $F^{\text{irr}}$  and  $\Lambda$ , equal to zero, this equation reduces to eq. (4.7) of paper I, which, as was discussed, yields the Liouville equation in the continuum limit. It may be verified (cf. also appendix A), using the explicit elements  $M_{\beta\gamma\delta,nn'}$  and the definition (4.10), that

$$\sum_{\substack{n',n'' \\ \gamma,\delta}} \sum_{\substack{\gamma',\delta' \\ \beta'}} \left( \frac{\partial M_{\beta\gamma\delta,nn'}}{\partial \alpha_{\beta',n''}} \right) M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n''n'''} = 0. \tag{4.20}$$

The Fokker-Planck equation (4.19) may therefore also be written as

$$\frac{\partial P(\{\alpha_n\}, t)}{\partial t} = \sum_n \sum_\beta \frac{\partial}{\partial \alpha_{\beta,n}} \left[ -F_{\beta,n}^{\text{rev}} - F_{\beta,n}^{\text{irr}} + \sum_{n'} \sum_{\beta'} \frac{\partial}{\partial \alpha_{\beta',n'}} D_{\beta\beta',nn'} \right] P(\{\alpha_n\}, t), \tag{4.21}$$

where the matrix of diffusion coefficients is given by

$$D_{\beta\beta',nn'} = \sum_{n''} \sum_{\substack{\gamma,\delta \\ \gamma',\delta' \\ \beta''}} M_{\beta\gamma\delta,nn''} M_{\beta'\gamma'\delta',n''n'''} \Lambda_{\gamma\delta\gamma'\delta',n''n'''}. \tag{4.22}$$

With the aid of eqs. (4.14) and (4.18), one finds the explicit expressions

$$D_{0\beta,nn'} = D_{\beta 0,nn'} = 0, \tag{4.23}$$

$$D_{ij,nn'} = D_{ji,n'n} = k\Delta^{-3} \left( L U \nabla_n \cdot \nabla_{n'} \delta_{nn'} + \left( L_v + \frac{1}{3} L \right) \nabla_n \nabla_{n'} \delta_{nn'} \right)_{ij}, \tag{4.24}$$

$$D_{i4,nn'} = D_{4i,n'n} = k\Delta^{-3} (2L \nabla_n \cdot \overline{\nabla_{n'} v_n} \delta_{nn'} + L_v \nabla_n \nabla_{n'} \cdot v_n \delta_{nn'})_i, \tag{4.25}$$

$$\begin{aligned} D_{44,nn'} = & k\Delta^{-3} L_q \nabla_n \cdot \nabla_{n'} \delta_{nn'} + 2k\Delta^{-3} \overline{L \nabla_n v_n} : \nabla_{n'} v_n \delta_{nn'} + k\Delta^{-3} L_v (\nabla_n \cdot v_n) \\ & \times (\nabla_{n'} \cdot v_n) \delta_{nn'}. \end{aligned} \tag{4.26}$$

Here  $U$  denotes the unit tensor, while the gradient operators act on everything behind them.

The form (4.21) of the Fokker-Planck equation would also have been obtained if the equations (4.11) had been interpreted as stochastic differential equations in the Itô sense<sup>14</sup>). This in fact would suffice to conclude that, if (4.11) is interpreted as a Stratonovich equation, the corresponding Itô equa-

tion is unmodified and vice versa, as was already mentioned at the beginning of this section. The conclusion therefore is that eq. (4.11) leads to unambiguous results, independent of its interpretation either in the Itô or Stratonovich sense. We stress, however, that stochastic equations for functions of the variables  $\rho$ ,  $j$  and  $e_v$ , as e.g. the entropy or the temperature\*, do not necessarily get the same form on the basis of the two interpretations of eq. (4.11), although their physical content remains of course the same.

We finally note that the Fokker–Planck equation (4.19) may also be written in the form

$$\frac{\partial P(\{\alpha_n\}, t)}{\partial t} = \sum_n \sum_\beta \left[ -F_{\beta,n}^{\text{rev}} \frac{\partial}{\partial \alpha_{\beta,n}} + \frac{\partial}{\partial \alpha_{\beta,n}} \left( -F_{\beta,n}^{\text{irr}} + \sum_{n'} \sum_{\beta'} D_{\beta\beta',nn'} \frac{\partial}{\partial \alpha_{\beta',n'}} \right) \right] P(\{\alpha_n\}, t). \quad (4.27)$$

Here use has been made of the fact that (cf. also eq. (4.8) of paper I)

$$\sum_\beta \frac{\partial F_{\beta,n}^{\text{rev}}}{\partial \alpha_{\beta,n}} = 0, \quad (4.28)$$

and that

$$\sum_{n'} \sum_{\beta'} \frac{\partial D_{\beta\beta',nn'}}{\partial \alpha_{\beta',n'}} = 0. \quad (4.29)$$

These equations immediately follow from eqs. (4.12) and (4.23)–(4.26), together with the definition (4.6) of the discrete gradient operator.

## 5. The form of the dissipative currents

Until now we have left the form of the viscous pressure tensor and the heat current undetermined and have only assumed that they are functions of the hydrodynamic variables. We shall now determine the form of these currents compatible with our assumption that the random currents are Gaussian processes with white noise. It is on the basis of this assumption that we were able to derive in the last section a Fokker–Planck equation for the distribution function  $P(\{\alpha_n\}, t)$  of the hydrodynamic variables. For these variables the property of microscopic reversibility holds<sup>12)</sup>. This implies that the fundamental solution of the Fokker–Planck equation  $P(\{\alpha_n\}, t | \{\alpha'_n\})$ , which gives the conditional probability density that the system is in the state  $\{\alpha_n\}$  at

\* In this connection, and also in relation to remarks made by Fox<sup>15)</sup>, we discuss in appendix B the equation for the temperature.

time  $t$  if it is initially in the state  $\{\alpha'_n\}$  at time zero, must satisfy the property

$$P(\{\alpha_n\}, t \mid \{\alpha'_n\})P^{eq}(\{\alpha'_n\}) = P(\{\tilde{\alpha}'_n\}, t \mid \{\tilde{\alpha}_n\})P^{eq}(\{\tilde{\alpha}_n\}). \quad (5.1)$$

Here  $\tilde{\alpha}_{\beta,n} = \epsilon_\beta \alpha_{\beta,n}$ , where  $\epsilon_\beta = 1$  if  $\alpha_{\beta,n}$  is even under time reversal and  $\epsilon_\beta = -1$  if  $\alpha_{\beta,n}$  is odd under time reversal, so that (cf. eq. (2.1))  $\tilde{\alpha}_{0,n} = \alpha_{0,n}$  and  $\tilde{\alpha}_{i,n} = -\alpha_{i,n}$  and  $\tilde{\alpha}_{4,n} = \alpha_{4,n}$ .

From property (5.1) and the Fokker-Planck equation (4.27) the following set of conditions may be derived:

$$\sum_{n,\beta} F_{\beta,n}^{rev} \frac{\partial P^{eq}}{\partial \alpha_{\beta,n}} = 0 \quad (5.2)$$

for the equilibrium distribution, which is eq. (I.4.9) for the ideal fluid if  $P^{eq}$  is inserted, and

$$F_{\beta,n}^{int} = \sum_{n'} \sum_{\beta'} D_{\beta\beta',nn'} \frac{\partial \ln P^{eq}}{\partial \alpha_{\beta',n'}}. \quad (5.3)$$

These are the so-called potential conditions<sup>16)</sup> as applied to the case considered here, in which eqs. (4.28) and (4.29) hold\*.

We will analyse eqs. (5.2) and (5.3) in the continuum limit, in which the discretized variables become fields again. In this limit, we have e.g.

$$\lim_{\Delta \rightarrow 0} F_{0,n}^{rev} = \lim_{\Delta \rightarrow 0} -\nabla_n \cdot j_n = -\nabla \cdot j(\mathbf{r}) = \{\alpha_0(\mathbf{r}), H\}. \quad (5.4)$$

Here, use has been made of eq. (4.12) and of the Poisson-bracket expressions defined in the first section (cf. eqs. (2.6) and (2.9)). Analogously, one finds for the other components of  $F_{\beta,n}^{rev}$

$$\lim_{\Delta \rightarrow 0} F_{\beta,n}^{rev} = \{\alpha_\beta(\mathbf{r}), H\}. \quad (5.5)$$

With the aid of this result, eq. (5.2) becomes in the continuum limit

$$\begin{aligned} \lim_{\Delta \rightarrow 0} \sum_n \sum_\beta F_{\beta,n}^{rev} \frac{\partial P^{eq}(\{\alpha_n\})}{\partial \alpha_{\beta,n}} &= \sum_\beta \int d\mathbf{r} \{\alpha_\beta(\mathbf{r}), H\} \frac{\delta P^{eq}(\{\alpha(\mathbf{r})\})}{\delta \alpha_\beta(\mathbf{r})} \\ &= \{P^{eq}(\{\alpha(\mathbf{r})\}), H\} = 0, \end{aligned} \quad (5.6)$$

provided that  $P^{eq}(\{\alpha_n\})$  converges to a proper functional  $P^{eq}(\{\alpha(\mathbf{r})\})$ . This equation expresses the fact that  $P^{eq}$  should be a stationary solution of the Liouville-equation. As discussed in paper I, the Einstein distribution,

$$P^{eq}(\{\alpha(\mathbf{r})\}) \sim e^{S(\{\alpha(\mathbf{r})\})/k}, \quad (5.7)$$

\* In the derivation of eqs. (5.2) and (5.3) use is made of the fact that  $F_{\beta}^{rev}(\{\alpha\}) = -\epsilon_\beta F_{\beta}^{rev}(\{\alpha\})$  (cf. eq. (4.12)) and of the property  $F_{\beta}^{int}(\{\alpha\}) = \epsilon_\beta F_{\beta}^{int}(\{\alpha\})$ .

where  $S$  is the total entropy, defined by

$$S(\{\alpha(\mathbf{r})\}) \equiv \int d\mathbf{r} s_v(\mathbf{r}), \quad (5.8)$$

is indeed a stationary solution. Similarly, the equilibrium distribution for a fluid in thermal contact with a heat bath at temperature  $T_0$ ,

$$P^{\text{eq}}(\{\alpha(\mathbf{r})\}) \sim e^{-(H-T_0S)/kT_0}, \quad (5.9)$$

and the equilibrium distribution for a fluid which can in addition exchange mass with a reservoir with chemical potential  $\mu_0$

$$P^{\text{eq}}(\{\alpha(\mathbf{r})\}) \sim e^{-(H-T_0S-\mu_0M)/kT_0}, \quad (5.10)$$

where  $M$  is the total mass of the fluid,

$$M \equiv \int d\mathbf{r} \rho(\mathbf{r}), \quad (5.11)$$

satisfy eq. (5.6).

We now turn to the evaluation of eq. (5.3). In the continuum limit this equation becomes

$$F_{\beta}^{\text{irr}}(\mathbf{r}) = \sum_{\beta'} \int d\mathbf{r}' D_{\beta\beta'}(\mathbf{r}, \mathbf{r}') \frac{\delta \ln P^{\text{eq}}(\{\alpha(\mathbf{r})\})}{\delta \alpha_{\beta'}(\mathbf{r}')}, \quad (5.12)$$

where we have defined

$$F_{\beta}^{\text{irr}}(\mathbf{r}) \equiv \lim_{\Delta \rightarrow 0} F_{\beta, \mathbf{n}}^{\text{irr}}, \quad (5.13)$$

$$D_{\beta\beta'}(\mathbf{r}, \mathbf{r}') = \lim_{\Delta \rightarrow 0} D_{\beta\beta', \mathbf{nn}'}. \quad (5.14)$$

In view of the definitions (4.13), we obtain for  $F_{\beta}^{\text{irr}}(\mathbf{r})$

$$F_0^{\text{irr}}(\mathbf{r}) = 0, \quad F_i^{\text{irr}}(\mathbf{r}) = -(\nabla \cdot \mathbf{II}^s)_i, \quad F_4^{\text{irr}} = -\nabla \cdot [\mathbf{II}^s \cdot \mathbf{v} + \mathbf{J}^s], \quad (5.15)$$

and, from eqs. (4.23)–(4.26), for the elements of  $D_{\beta\beta'}(\mathbf{r}, \mathbf{r}')$

$$D_{0\beta}(\mathbf{r}, \mathbf{r}') = D_{\beta 0}(\mathbf{r}, \mathbf{r}') = 0, \quad (5.16)$$

$$D_{ij}(\mathbf{r}, \mathbf{r}') = k \left( L \mathbf{U} \mathbf{V} \cdot \nabla' \delta(\mathbf{r} - \mathbf{r}') + \left( L_v + \frac{1}{3} L \right) \nabla \nabla' \delta(\mathbf{r} - \mathbf{r}') \right)_{ij}, \quad (5.17)$$

$$D_{i4}(\mathbf{r}, \mathbf{r}') = D_{4i}(\mathbf{r}', \mathbf{r}) = k (2L \nabla \cdot \overline{\nabla' \mathbf{v}(\mathbf{r}')}) \delta(\mathbf{r} - \mathbf{r}') + L_v \nabla (\nabla' \cdot \mathbf{v}(\mathbf{r}')) \delta(\mathbf{r} - \mathbf{r}'), \quad (5.18)$$

$$D_{44}(\mathbf{r}, \mathbf{r}') = k L_q \nabla \cdot \nabla' \delta(\mathbf{r} - \mathbf{r}') + 2k L \overline{\nabla \mathbf{v}(\mathbf{r})} : \nabla' \mathbf{v}(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') + k L_v (\nabla \cdot \mathbf{v}(\mathbf{r})) (\nabla' \cdot \mathbf{v}(\mathbf{r}')) \delta(\mathbf{r} - \mathbf{r}'). \quad (5.19)$$

One immediately sees that eq. (5.12) is trivially fulfilled for  $\beta = 0$ . In order to evaluate this equation for other values of  $\beta$ , we also need the functional derivatives of the logarithm of the equilibrium distribution. For the case of a fluid which is materially and energetically isolated, to which the equilibrium distribution (5.7) applies, these are

$$\frac{\delta \ln P^{\text{eq}}}{\delta \alpha_i(\mathbf{r})} = -k^{-1} \frac{v_i(\mathbf{r})}{T(\mathbf{r})}, \quad \frac{\delta \ln P^{\text{eq}}}{\delta \alpha_4(\mathbf{r})} = k^{-1} \frac{1}{T(\mathbf{r})}. \quad (5.20)$$

Substitution of eqs. (5.15)–(5.20) into eq. (5.12) yields

$$F_i^{\text{irr}}(\mathbf{r}) = -(\nabla \cdot \Pi^s(\mathbf{r}))_i = \left( 2L\nabla \cdot \frac{\overline{\nabla \mathbf{v}(\mathbf{r})}}{T(\mathbf{r})} + L_v \nabla \frac{\nabla \cdot \mathbf{v}}{T(\mathbf{r})} \right)_i, \quad (5.21)$$

$$\begin{aligned} F_4^{\text{irr}}(\mathbf{r}) &= -\nabla \cdot (\Pi^s(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r}) + \mathbf{J}^s(\mathbf{r})) \\ &= -L_q \nabla^2 \frac{1}{T(\mathbf{r})} + L\nabla \cdot \left( \frac{\overline{\nabla \mathbf{v}(\mathbf{r})}}{T(\mathbf{r})} \cdot \mathbf{v}(\mathbf{r}) \right) + L_v \nabla \cdot \left( \frac{\nabla \cdot \mathbf{v}(\mathbf{r})}{T(\mathbf{r})} \mathbf{v}(\mathbf{r}) \right). \end{aligned} \quad (5.22)$$

Eqs. (5.21) and (5.22) lead to the identification

$$\Pi^s(\mathbf{r}) = -2L \frac{\overline{\nabla \mathbf{v}(\mathbf{r})}}{T(\mathbf{r})} - L_v \mathbf{U} \frac{\nabla \cdot \mathbf{v}(\mathbf{r})}{T(\mathbf{r})}, \quad (5.23)$$

$$\mathbf{J}^s(\mathbf{r}) = L_q \nabla \frac{1}{T(\mathbf{r})} = -\frac{L_q}{T^2(\mathbf{r})} \nabla T(\mathbf{r}). \quad (5.24)$$

We have thus shown that the only expressions for the phenomenological laws compatible with the assumptions of Gaussian white noise for the random currents are those given by eqs. (5.23) and (5.24). These laws have the usual form of the phenomenological laws<sup>12)</sup> in the sense that the dissipative currents are *linear in the gradients* of the state variables. It is, however, important to realize that in these expressions the Onsager coefficients  $L$ ,  $L_v$  and  $L_q$  may only depend on the equilibrium quantities  $\rho_0$  and  $T_0$  and *not* on the fluctuating fields. As a consequence, the shear viscosity  $\eta$  and the bulk viscosity  $\eta_v$ , which are given by

$$\eta = \frac{L}{T(\mathbf{r})}, \quad \eta_v = \frac{L_v}{T(\mathbf{r})}, \quad (5.25)$$

are within the present scheme proportional to the inverse of the temperature, while the heat conductivity  $\lambda$ , given by

$$\lambda = \frac{L_q}{T^2(\mathbf{r})}, \quad (5.26)$$

is proportional to  $T^{-2}$ . In a real system, such temperature dependences of the transport coefficients do not occur. This implies that the assumption of

*Gaussian* white noise is, strictly speaking, not compatible with the physical phenomenological laws. The present scheme nevertheless is approximately valid if the temperature fluctuations may be considered to be sufficiently small so that the dissipative currents (5.23) and (5.24) can be linearized completely in the fluctuating fields\*.

In the above analysis, use has been made of the Einstein formula (5.7) for the equilibrium distribution function. One easily checks however, that substitution of the equilibrium distributions (5.9) and (5.10) into eq. (5.12) leads to the same conclusions.

Even though our final results have been derived in the continuum limit, we emphasize the necessity of the discretization procedure for intermediate calculations. This is illustrated by the fact that, if one would immediately go over to the continuum limit in the Fokker–Planck equation (4.27), one of the terms arising would read

$$\sum_{\beta\gamma} \int d\mathbf{r} \int d\mathbf{r}' \frac{\delta D_{\beta\gamma}(\mathbf{r}, \mathbf{r}')}{\delta \alpha_{\beta}(\mathbf{r})} \frac{\delta P}{\delta \alpha_{\gamma}(\mathbf{r}')}. \quad (5.27)$$

The interpretation of the functional derivative

$$\frac{\delta D_{\beta\gamma}(\mathbf{r}, \mathbf{r}')}{\delta \alpha_{\beta}(\mathbf{r})} \quad (5.28)$$

in this expression is not clear, however, due to the delta-function  $\delta(\mathbf{r} - \mathbf{r}')$  in  $D_{\beta\gamma}(\mathbf{r}, \mathbf{r}')$  (cf. eqs. (5.16)–(5.19)). This ambiguity has been avoided by the discretization procedure. For a more elaborate discussion of the discretization rules, especially in relation with the proper choice of variables, we refer to paper I. In this connection it may be noted that the difficulties encountered in a Fokker–Planck equation for fields are related to the well-known fact that equilibrium mean square fluctuations of state variables of a volume element are inversely proportional to the size of the volume element. In the continuum limit, such quantities diverge.

We finally remark that the quadratic term  $\mathbf{II} \cdot \mathbf{v}$ , occurring in the energy-equation, has given rise to controversy. It has been argued<sup>15)</sup> that this term, which plays an essential role in our analysis, is the origin of an inherent inconsistency of non-linear fluctuation theory. In appendix B we show, however, that this criticism is based on an erroneous argument.

\* In fact, according to eqs. (5.25) and (5.26), the transport coefficients must also be considered to be independent of the fluctuating density and to be only functions of  $\rho_0$ . Again this will in general only be legitimate in as far as density fluctuations are sufficiently small.

## 6. The linear response to an external force

In this section we will study the linear response to a (small) external force density. In the presence of an external potential  $V(\mathbf{r}, t)$ , the hydrodynamic equations (3.1)–(3.3) become

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}, \quad (6.1)$$

$$\frac{\partial \mathbf{j}}{\partial t} = -\nabla \cdot [\mathbf{v}\mathbf{j} + \mathbf{p} + \mathbf{\Pi}] - \rho \nabla V, \quad (6.2)$$

$$\frac{\partial e_v}{\partial t} = -\nabla \cdot [\mathbf{v}(e_v + p) + \mathbf{v} \cdot \mathbf{\Pi} + \mathbf{J}] - \mathbf{j} \cdot \nabla V, \quad (6.3)$$

where, as before,  $e_v$  is the total energy density in the absence of the force, i.e.  $e_v = \frac{1}{2}\mathbf{j}^2/\rho + u_v$ .

We will analyse, along the lines of ref. 17, the linear response of the system to this potential using the Fokker–Planck equation equivalent with the above equations. As was discussed in the preceding sections, the Fokker–Planck equation should in principle be derived starting from a set of discretized equations. For conciseness, we will in this section not explicitly perform this program, but write the equations in a continuum-notation throughout. One may easily verify, however, that a more careful analysis, based on the discretization rules, leads to the same results.

As the terms in eqs. (6.2) and (6.3) arising from the external potential are of a purely mechanical nature, they already appear in the equations of motion of the ideal fluid. In the presence of the potential, the Hamiltonian  $H$  is given by

$$H(t) = H_0 + H_1(t), \quad (6.4)$$

with

$$H_0 \equiv \int d\mathbf{r} e_v(\mathbf{r}), \quad H_1(t) \equiv \int d\mathbf{r} \rho(\mathbf{r}) V(\mathbf{r}, t). \quad (6.5)$$

The equations of motion (6.1)–(6.3) for the ideal fluid (i.e.  $\mathbf{\Pi} = 0$ ,  $\mathbf{J} = 0$ ) may with the aid of this Hamiltonian be written as

$$\frac{\partial \alpha(\mathbf{r}, t)}{\partial t} = \{\alpha(\mathbf{r}, t), H(t)\} = \{\alpha(\mathbf{r}, t), H_0\} + \{\alpha(\mathbf{r}, t), H_1(t)\}. \quad (6.6)$$

This may be verified, using the fact that

$$\frac{\delta H_0}{\delta \alpha_\beta(\mathbf{r})} = \delta_{\beta 4}, \quad \frac{\delta H_1}{\delta \alpha_\beta(\mathbf{r})} = V(\mathbf{r}, t) \delta_{\beta 0}, \quad (6.7)$$

so that eq. (6.6) becomes

$$\frac{\partial \alpha_{\beta}(\mathbf{r}, t)}{\partial t} = \int d\mathbf{r}' L_{\beta 4}(\mathbf{r}, \mathbf{r}') + \int d\mathbf{r}' L_{\beta 0}(\mathbf{r}, \mathbf{r}') V(\mathbf{r}', t). \quad (6.8)$$

Upon substitution of the explicit expressions (2.3) for the elements  $L_{\beta 4}$  and  $L_{\beta 0}$  the ideal fluid equations immediately follow.

The Liouville equation now becomes\*

$$\begin{aligned} \frac{\partial P(\{\alpha(\mathbf{r})\}, t)}{\partial t} &= \{H(t), P\} \equiv L_0 P + L_1(t) P, \\ &= \int d\mathbf{r} \int d\mathbf{r}' \sum_{\beta=0}^4 [L_{4\beta}(\mathbf{r}, \mathbf{r}') + V(\mathbf{r}, t) L_{0\beta}(\mathbf{r}, \mathbf{r}')] \frac{\delta P}{\delta \alpha_{\beta}(\mathbf{r}')}, \end{aligned} \quad (6.9)$$

where  $L_0$  is the Liouville-operator in the absence of the external force and  $L_1(t)$  the contribution to the total Liouville operator due to the external force,

$$L_0 P \equiv \{H_0, P\}, \quad L_1(t) P \equiv \{H_1(t), P\}. \quad (6.10)$$

The definition of the Poisson brackets given in paper I, eq. (I. 3.23), immediately implies that  $\{H(t), S\} = 0$ . Consequently

$$(L_0 + L_1(t)) e^{-(H(t)-T_0 S)/kT_0} = \{H(t), e^{-(H(t)-T_0 S)/kT_0}\} = 0. \quad (6.11)$$

For  $V = 0$ , this equation reduces to eq. (5.6) for the equilibrium distribution (5.9). By expanding eq. (6.11) up to linear order in  $V$ , one obtains

$$L_1(t) P^{\text{eq}} = \frac{1}{kT_0} L_0 \left( \int d\mathbf{r} \rho(\mathbf{r}) V(\mathbf{r}, t) P^{\text{eq}} \right) = \frac{1}{kT_0} \int d\mathbf{r} \nabla \cdot \mathbf{j}(\mathbf{r}) V(\mathbf{r}, t) P^{\text{eq}}. \quad (6.12)$$

Here  $P^{\text{eq}}$  is the equilibrium distribution of the system in the absence of the external potential, which satisfies  $L_0 P^{\text{eq}} = 0$ .

We now return to the general case of a fluid with dissipation in the presence of an external potential. Instead of the Liouville equation (6.9), we then find for the evolution of the probability distribution a Fokker-Planck equation. Along similar lines as in the previous section one may show, on the basis of the fact that the correlation functions of the random currents are not modified by the external force, that the Fokker-Planck equation becomes

$$\frac{\partial P(\{\alpha(\mathbf{r})\}, t)}{\partial t} = (L_0 + L_1(t) + M) P(\{\alpha(\mathbf{r})\}, t), \quad (6.13)$$

where  $M$  may formally be defined as

$$M \equiv - \int d\mathbf{r} \sum_{\beta} \frac{\delta}{\delta \alpha_{\beta}(\mathbf{r})} F_{\beta}^{\text{irr}}(\mathbf{r}) + \int d\mathbf{r} \int d\mathbf{r}' \sum_{\beta\beta'} \frac{\delta}{\delta \alpha_{\beta}(\mathbf{r})} D_{\beta\beta'}(\mathbf{r}, \mathbf{r}') \frac{\delta}{\delta \alpha_{\beta'}(\mathbf{r}')}. \quad (6.14)$$

\* Again the incompressibility of the flow in phase space is easily checked using the discretization rules.



A more precise definition of  $M$  should be given in the context of the discrete formulation, cf. section 4, in particular eq. (4.27).

If we write  $P$  as

$$P(\{\alpha(r)\}, t) = P^{eq}(\{\alpha(r)\}) + P^1(\{\alpha(r)\}, t), \quad (6.15)$$

where  $P^1$  is the deviation from  $P^{eq}$ , one obtains from eq. (6.13) up to linear order in the external potential

$$\frac{\partial P^1(\{\alpha(r)\}, t)}{\partial t} = (L_0 + M)P^1(\{\alpha(r)\}, t) + L_1(t)P^{eq}(\{\alpha(r)\}). \quad (6.16)$$

We will assume that  $V(r, t) \rightarrow 0$  for  $t \rightarrow -\infty$  and that the system was initially in equilibrium. With the aid of eq. (6.12) the formal solution of eq. (6.16) can then be written as

$$P^1(\{\alpha(r)\}, t) = \frac{1}{kT_0} \int_{-\infty}^t dt' \int dr \exp\{(L_0 + M)(t - t')\} \nabla \cdot j(r) \times V(r, t) P^{eq}(\{\alpha(r)\}). \quad (6.17)$$

From this equation one easily calculates the average linear response to the perturbation. The average  $\langle \alpha(r, t) \rangle$  is defined by

$$\begin{aligned} \langle \alpha(r, t) \rangle &= \lim_{\Delta \rightarrow 0} \int \prod_n d\alpha_n \alpha_n P(\{\alpha_n\}, t) \\ &\equiv \iint d\{\alpha(r)\} \alpha(r) P(\{\alpha(r)\}, t) \end{aligned} \quad (6.18)$$

and the equilibrium correlation functions  $\langle \alpha(r, t) \alpha(r', 0) \rangle_{eq}$  by

$$\begin{aligned} \langle \alpha(r, t) \alpha(r', 0) \rangle_{eq} &\equiv \iint d\{\alpha(r)\} \iint d\{\alpha'(r)\} \alpha(r) \alpha'(r') P(\{\alpha(r)\}, t \mid \{\alpha'(r)\}) \\ &\quad \times P^{eq}(\{\alpha'(r)\}), \end{aligned} \quad (6.19)$$

where, as before,  $P(\{\alpha(r)\}, t \mid \{\alpha'(r)\})$  is the fundamental solution of the Fokker-Planck equation in the absence of the potential.

From eq. (6.17), we obtain for

$$\begin{aligned} \Delta \langle \rho(r, t) \rangle &\equiv \langle \rho(r, t) \rangle - \langle \rho(r, t) \rangle_{eq}, \\ \Delta \langle \rho(r, t) \rangle &= \frac{1}{kT_0} \int_{-\infty}^t dt' \int dr' \left[ \iint d\{\alpha(r)\} \rho(r) \exp\{(L_0 + M)(t - t')\} \right. \\ &\quad \left. \times \nabla' \cdot j(r') P^{eq}(\{\alpha(r)\}) \right] V(r', t'). \end{aligned} \quad (6.20)$$

To proceed, we note that

$$P(\{\boldsymbol{\alpha}(\mathbf{r})\}, 0 \mid \{\boldsymbol{\alpha}'(\mathbf{r}')\}) = \lim_{\Delta \rightarrow 0} \prod_n \delta(\boldsymbol{\alpha}_n - \boldsymbol{\alpha}'_n) \equiv \delta(\{\boldsymbol{\alpha}(\mathbf{r})\} - \{\boldsymbol{\alpha}'(\mathbf{r}')\}). \quad (6.21)$$

With the aid of this initial condition, we may rewrite the term between square brackets in eq. (6.20), viz.

$$\begin{aligned} & \int d\{\boldsymbol{\alpha}(\mathbf{r})\} \rho(\mathbf{r}) \exp\{(\mathbb{L}_0 + \mathbb{M})(t - t')\} \nabla' \cdot \mathbf{j}(\mathbf{r}') P^{\text{eq}}(\{\boldsymbol{\alpha}(\mathbf{r})\}) \\ &= \int d\{\boldsymbol{\alpha}(\mathbf{r})\} \int d\{\boldsymbol{\alpha}'(\mathbf{r}')\} \rho(\mathbf{r}) \exp\{(\mathbb{L}_0 + \mathbb{M})(t - t')\} \nabla' \cdot \mathbf{j}'(\mathbf{r}') \\ & \quad \times P(\{\boldsymbol{\alpha}(\mathbf{r})\}, 0 \mid \{\boldsymbol{\alpha}'(\mathbf{r}')\}) P^{\text{eq}}(\{\boldsymbol{\alpha}'(\mathbf{r}')\}) \\ &= \int d\{\boldsymbol{\alpha}(\mathbf{r})\} \int d\{\boldsymbol{\alpha}'(\mathbf{r}')\} \rho(\mathbf{r}) \nabla' \cdot \mathbf{j}'(\mathbf{r}') P(\{\boldsymbol{\alpha}(\mathbf{r})\}, t - t' \mid \{\boldsymbol{\alpha}'(\mathbf{r}')\}) \\ & \quad \times P^{\text{eq}}(\{\boldsymbol{\alpha}'(\mathbf{r}')\}) \\ &= \langle \rho(\mathbf{r}, t) \nabla' \cdot \mathbf{j}(\mathbf{r}', t') \rangle_{\text{eq}}. \end{aligned} \quad (6.22)$$

In the above, use was made of the definition (6.19) of equilibrium correlation functions and of the fact that the fundamental solution of the Fokker-Planck equation in the  $V = 0$  case is formally given by

$$P(\{\boldsymbol{\alpha}(\mathbf{r})\}, t \mid \{\boldsymbol{\alpha}'(\mathbf{r}')\}) = \exp\{(\mathbb{L}_0 + \mathbb{M})t\} P(\{\boldsymbol{\alpha}(\mathbf{r})\}, 0 \mid \{\boldsymbol{\alpha}'(\mathbf{r}')\}). \quad (6.23)$$

Substitution of eq. (6.22) into eq. (6.20) yields

$$\begin{aligned} \Delta \langle \rho(\mathbf{r}, t) \rangle &= \frac{1}{kT_0} \int_{-\infty}^t dt' \int d\mathbf{r}' \langle \rho(\mathbf{r}, t) \nabla' \cdot \mathbf{j}(\mathbf{r}', t') \rangle_{\text{eq}} V(\mathbf{r}', t') \\ &= \int_{-\infty}^{\infty} dt' \int d\mathbf{r}' \mathbf{G}_{\rho j}(\mathbf{r} - \mathbf{r}', t - t') \cdot \mathbf{K}(\mathbf{r}', t'). \end{aligned} \quad (6.24)$$

Here  $\mathbf{K}(\mathbf{r}, t) = -\nabla V(\mathbf{r}, t)$  is the force per unit mass due to the external potential, and  $\mathbf{G}_{\rho j}$  a Green's function defined by

$$\mathbf{G}_{\rho j}(\mathbf{r} - \mathbf{r}', t - t') \equiv \frac{1}{kT_0} \langle \rho(\mathbf{r}, t) \mathbf{j}(\mathbf{r}', t') \rangle_{\text{eq}} \Theta(t - t'). \quad (6.25)$$

In this definition,  $\Theta(t - t')$  is the Heavyside step function.

One similarly obtains

$$\Delta \langle \mathbf{j}(\mathbf{r}, t) \rangle = \int_{-\infty}^{+\infty} dt' \int d\mathbf{r}' \mathbf{G}_{j j}(\mathbf{r} - \mathbf{r}', t - t') \cdot \mathbf{K}(\mathbf{r}', t'), \quad (6.26)$$

and

$$\Delta \langle e_v(\mathbf{r}, t) \rangle = \int_{-\infty}^{+\infty} dt' \int d\mathbf{r}' \mathbf{G}_{ej}(\mathbf{r} - \mathbf{r}', t - t') \cdot \mathbf{K}(\mathbf{r}', t'), \quad (6.27)$$

with

$$\mathbf{G}_{ij}(\mathbf{r} - \mathbf{r}', t - t') \equiv \frac{1}{kT_0} \langle \mathbf{j}(\mathbf{r}, t) \mathbf{j}(\mathbf{r}', t') \rangle_{\text{eq}} \Theta(t - t') \quad (6.28)$$

and

$$\mathbf{G}_{ej}(\mathbf{r} - \mathbf{r}', t - t') \equiv \frac{1}{kT_0} \langle e_v(\mathbf{r}, t) \mathbf{j}(\mathbf{r}', t') \rangle_{\text{eq}} \Theta(t - t'). \quad (6.29)$$

Expressions (6.24)–(6.29) represent the fluctuation dissipation theorem for the present case. The equilibrium correlation functions occurring in these theorems and giving the *linear* response to the external force, should be calculated using the *nonlinear* equations describing the equilibrium fluctuations of the fluid. These non-linear terms in the equations of motion lead to the so-called mode–mode coupling contributions to the above correlation functions. One obtains in that way within the framework of a hydrodynamic fluctuation theory on the one hand the long time tails of these functions and on the other hand the mode–mode coupling expressions for the renormalized transport coefficients which are of special interest in particular near the critical point.

We finally note that Ma<sup>18)</sup> derived a fluctuation-dissipation theorem similar to those given above for a somewhat more simple system obeying a non-linear stochastic differential equation using a graph-theoretical procedure.

### Appendix A

In this appendix we show that the discretized hydrodynamic eqs. (4.1)–(4.5) may either be interpreted in the Itô or in the Stratonovich sense. As discussed e.g. by Arnold<sup>14)</sup>, the Stratonovich equations

$$(S) \quad \frac{da_i}{dt} = F_i + \sum_j M_{ij} f_j, \quad (A.1)$$

where  $F_i$  and  $M_{ij}$  depend on the  $a_i$  and where  $f_i$  is a random force with correlations

$$\overline{f_i(t) f_j(t')} = 2\Lambda_{ij} \delta(t - t'), \quad (A.2)$$

is equivalent to the Itô equation

$$(I) \quad \frac{da_i}{dt} = F_i + \sum_j \left( \frac{\partial M_{ij}}{\partial a_k} \right) M_{kl} \Lambda_{ji} + \sum_j M_{ij} f_j. \tag{A.3}$$

Generalization to the case considered in this paper shows that the discretized Stratonovich stochastic hydrodynamic equations written in the form (4.11),

$$(S) \quad \frac{\partial \alpha_{\beta,n}}{\partial t} = F_{\beta,n}^{\text{rev}} + F_{\beta,n}^{\text{irr}} + \sum_n \sum_{\gamma,\delta=0}^4 M_{\beta\gamma\delta,nn'} f_{\gamma\delta,n'}, \tag{A.4}$$

are equivalent with the Itô equations

$$(I) \quad \frac{\partial \alpha_{\beta,n}}{\partial t} = F_{\beta,n}^{\text{rev}} + F_{\beta,n}^{\text{irr}} + \sum_{n''} \sum_{\gamma,\delta=0}^4 M_{\beta\gamma\delta,nn''} f_{\gamma\delta,n''} \\ + \sum_{n''} \sum_{\gamma'\delta'} \left( \frac{\partial M_{\beta\gamma\delta,nn''}}{\partial \alpha_{\beta',n''}} \right) M_{\beta'\gamma'\delta',n''n''} \Lambda_{\gamma\delta\gamma'\delta',n''n''}. \tag{A.5}$$

Hence, if

$$\sum_{\substack{n'' \\ n''}} \sum_{\substack{\gamma\delta \\ \gamma'\delta'}} \left( \frac{\partial M_{\beta\gamma\delta,nn''}}{\partial \alpha_{\beta',n''}} \right) M_{\beta'\gamma'\delta',n''n''} \Lambda_{\gamma\delta\gamma'\delta',n''n''} = 0, \tag{A.6}$$

the Itô and Stratonovich equations are identical\*.

To show that eq. (A.6) is indeed obeyed, we notice that the only elements of  $M_{\beta\gamma\delta,nn'}$  which depend on the variables are the elements  $M_{4kl}$  (cf. eqs. (4.14)). Therefore, eq. (A.6) is trivially fulfilled for  $\beta = 0, 1, 2, 3$ . For  $\beta = 4$  one gets, using eqs. (4.14) and (4.17),

$$\sum_{\substack{n'' \\ n''}} \sum_{\substack{\gamma\delta \\ \gamma'\delta'}} \left( \frac{\partial M_{4\gamma\delta,nn''}}{\partial \alpha_{\beta',n''}} \right) M_{\beta'\gamma'\delta',n''n''} \Lambda_{\gamma\delta\gamma'\delta',n''n''} \\ = \frac{k}{\Delta^3} \sum_{\substack{n'' \\ n''}} \sum_{\substack{ikl \\ mp}} \left( \frac{\partial M_{4kl,nn''}}{\partial j_{i,n''}} \right) M_{imp,n''n''} L_{klmp} \delta_{n''n''} \\ = \frac{k}{\Delta^3} \sum_{\substack{n'' \\ n''}} \sum_{\substack{ikl \\ mp}} \delta_{ip} \delta_{il} L_{klmp} \frac{1}{\rho_{n''}} (\nabla_{k,n''} \delta_{nn''}) (\nabla_{m,n''} \delta_{n''n''}) \delta_{n''n''} \delta_{n''n''} \\ = 0. \tag{A.7}$$

The last step follows immediately from the fact that

$$(\nabla_{m,n''} \delta_{n''n''}) \delta_{n''n''} \delta_{n''n''} = 0, \tag{A.8}$$

\* The equivalence of the Itô and Stratonovich equations may of course also be established via the Fokker-Planck equation, cf. eq. (4.20) and the discussion after eq. (4.26).

since the term between brackets is only non-zero if  $n''' = n'' \pm \hat{e}_m$ , while the two other Kronecker delta's are only non-zero if  $n'' = n'''$ . Eq. (A.7) together with (A.6) establishes the equivalence of the Itô and Stratonovich equations in our case.

## Appendix B

In this appendix, we analyse the equations which led Fox to conclude that an "approach [similar to the one presented in this paper] to non-linear hydrodynamic fluctuations should be doubted and perhaps discarded"<sup>15</sup>.

Consider the equation for the fluctuating temperature of the fluid

$$\rho c_v \frac{dT}{dt} = -T \left( \frac{\partial p}{\partial T} \right)_\rho \nabla \cdot \mathbf{v} - \Pi^s : \nabla \mathbf{v} - \Pi^R : \nabla \mathbf{v} - \nabla \cdot \mathbf{J}^s - \nabla \cdot \mathbf{J}^R. \quad (\text{B.1})$$

Here  $c_v$  is the specific heat at constant volume of the fluid, and  $d/dt$  is the total time derivative, defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla. \quad (\text{B.2})$$

Upon linearization of eq. (B.1) one should, according to Fox, retain the term  $\Pi^R : \nabla \mathbf{v}$  since, even though it is bilinear in  $\Pi^R$  and  $\mathbf{v}$ , it is linear in  $\mathbf{v}$ , whereas the term  $\Pi^s : \nabla \mathbf{v}$  should be neglected since  $\Pi^s$  is of the order of  $\mathbf{v}$ , so that the whole term is of the order of  $\mathbf{v}^2$ . The author then proceeds to show that

$$\langle \Pi^R : \nabla \mathbf{v} \rangle_{\text{eq}} = \infty, \quad (\text{B.3})$$

on the basis of which result he concludes that the theory is inconsistent. In the above formula  $\langle \dots \rangle_{\text{eq}}$  denotes an average over an equilibrium ensemble.

The above argument is however incorrect. Indeed, the total pressure tensor is a macroscopic variable which is even under time reversal (i.e. even in the velocities of the constitutive particles in a molecular description). This follows from the fact that this quantity is defined through the momentum conservation equation (3.2). As a consequence its equal time correlation function with the velocity (gradient), which is odd in the particle velocities, vanishes (see e.g. ref. 12). Therefore, using the decomposition of  $\Pi$  into  $\Pi^s$  and  $\Pi^R$  given by eq. (3.4), one has

$$\langle \Pi(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle_{\text{eq}} = \langle \Pi^s(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle_{\text{eq}} + \langle \Pi^R(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle_{\text{eq}} = 0. \quad (\text{B.4})$$

Eq. (B.4) states that in equilibrium there is no heat production on the average. Since it follows from the phenomenological laws that

$$\Pi^s(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) < 0, \quad (\text{B.5})$$

one finds by combining eqs. (B.4) and (B.5)

$$\langle \Pi^R(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle_{\text{eq}} > 0. \quad (\text{B.6})$$

This inequality shows that the average temperature will decrease, if we neglect the term  $\Pi^s : \nabla \mathbf{v}$  in eq. (B.1) while retaining the term  $\Pi^R : \nabla \mathbf{v}$ . The above equations also show that the so-called ‘‘bilinear term’’  $\Pi^R : \nabla \mathbf{v}$  is on the average of the same order of magnitude as the ‘‘quadratic term’’  $\Pi^s : \nabla \mathbf{v}$  and that is inconsistent to retain one and neglect the other.

In passing we remark that eq. (B.1) is only correct if interpreted as a Stratonovich equation, since in the derivation of eq. (B.1) from eqs. (3.1)–(3.3) the usual rules for the transformation of variables have been applied to the stochastic equations. These rules, however, do not apply to Itô equations. The inconsistency of eq. (B.1) as an Itô equation is already apparent from the fact that in Itô calculus

$$\text{It}\hat{\circ}: \langle \Pi^R(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle_{\text{eq}} = 0, \quad (\text{B.7})$$

which is in contradiction with eq. (B.4).

We also note that expressions like  $\langle \Pi^R(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}', t') \rangle$  are discontinuous at  $t = t'$ . In view of the Stratonovich interpretation needed here, the equal time correlation  $\langle \Pi^R(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle$  appearing in eq. (B.4) is *defined* by

$$\langle \Pi^R(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{2} \langle \Pi^R(\mathbf{r}, t + \epsilon) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle + \frac{1}{2} \langle \Pi^R(\mathbf{r}, t - \epsilon) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle. \quad (\text{B.8})$$

We shall now verify by a direct calculation that eq. (B.4) also follows without the explicit use of the more elegant argument of time-reversal symmetry. This is done for the following set of equations

$$\nabla \cdot \mathbf{v} = 0, \quad (\text{B.9})$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \nabla \cdot \Pi^s - \nabla \cdot \Pi^R, \quad (\text{B.10})$$

$$\rho_0 c_{v_0} \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{J}^s - \nabla \cdot \mathbf{J}^R - \Pi^s : \nabla \mathbf{v} - \Pi^R : \nabla \mathbf{v}, \quad (\text{B.11})$$

where  $\mathbf{J}^s$  and  $\Pi^s$  now obey the linearized phenomenological laws (3.10) and (3.11), and where the correlations of  $\Pi^R$  and  $\mathbf{J}^R$  are given by eqs. (3.6)–(3.9), together with eq. (3.12). From here on, equilibrium quantities are indicated by a subscript 0. Apart from the term  $\Pi^s : \nabla \mathbf{v}$ , eqs. (B.9)–(B.11) are the equations considered by Fox, specialized to the case of an incompressible fluid.

We define the Fourier transform of a function  $f(\mathbf{r}, t)$  by

$$f(\mathbf{k}, \omega) = \int_{-\infty}^{+\infty} d\mathbf{r} \int_{-\infty}^{+\infty} dt e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} f(\mathbf{r}, t). \quad (\text{B.12})$$

After substitution of eq. (3.11) into eq. (B.10) and Fourier-transformation, we obtain for the velocity fluctuations generated by the random stress tensor in an aged system

$$\mathbf{v}(\mathbf{k}, \omega) = \frac{1}{i\omega\rho_0 + \eta k^2} i\mathbf{k} \cdot \mathbf{\Pi}^R(\mathbf{k}, \omega) \cdot (\mathbf{U} - \mathbf{k}\mathbf{k}/k^2), \quad (\text{B.13})$$

where use has been made of the fact that  $\mathbf{k} \cdot \mathbf{v}(\mathbf{k}, \omega) = 0$ . The equilibrium-correlations of  $\mathbf{\Pi}^R(\mathbf{k}, \omega)$  are given by

$$\langle \Pi_{ij}^R(\mathbf{k}, \omega) \Pi_{kl}^R(\mathbf{k}', \omega') \rangle_{\text{eq}} = 2kL_{ijkl}(2\pi)^4 \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega'). \quad (\text{B.14})$$

This equation follows from eq. (3.7) for an aged system in which the initial conditions for the hydrodynamic fields have been removed to time  $t_0 \rightarrow -\infty$ . With the aid of these equations, one finds

$$\begin{aligned} \langle \Pi^S(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle &= 2\eta \int \frac{d\mathbf{k} d\mathbf{k}' d\omega d\omega'}{(2\pi)^8} \overline{\langle \mathbf{k}\mathbf{v}(\mathbf{k}, \omega) : \mathbf{k}'\mathbf{v}(\mathbf{k}', \omega') \rangle}_{\text{eq}} \\ &\quad \times e^{-i(\mathbf{k}+\mathbf{k}') \cdot \mathbf{r} + i(\omega+\omega')t} \\ &= - \int \frac{d\mathbf{k} d\omega}{(2\pi)^4} \frac{4kT_0\eta^2 k^4}{\omega^2 \rho_0^2 + \eta^2 k^4}, \end{aligned} \quad (\text{B.15})$$

and

$$\langle \Pi^R(\mathbf{r}, t) : \nabla \mathbf{v}(\mathbf{r}, t) \rangle_{\text{eq}} = \int \frac{d\mathbf{k} d\omega}{(2\pi)^4} \frac{4kT_0\eta k^2}{-i\omega\rho_0 + \eta k^2} = \int \frac{d\mathbf{k} d\omega}{(2\pi)^4} \frac{4kT_0\eta^2 k^4}{\omega^2 \rho_0^2 + \eta^2 k^4}. \quad (\text{B.16})$$

Comparison of eqs. (B.15) and (B.16) shows indeed that eq. (B.4) is obeyed even though the integrals are divergent if no wave-vector cut-off, necessary to tame divergencies inherent to a continuum description, is introduced.

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