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# DIFFERENTIAL REAL SPACE RENORMALIZATION: THE LINEAR ISING CHAIN

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The differential real space renormalization method, recently introduced by Hilhorst et al., is applied to the linear Ising chain. It is shown that chains with spatially homogeneous as well as inhomogeneous or quenched random interactions can be treated. For the first two cases the free energy is computed by renormalization. The discussion includes also the case with a magnetic field, higher order interactions and the behavior of correlation functions under renormalization.

## 1. Introduction

In the renormalization theory of critical phenomena and phase transitions few examples exist which permit an exact and complete analysis. Recently Hilhorst et al.[1]) (hereafter to be referred to as HSL) derived an exact renormalization scheme for the $d = 2$-dimensional Ising model. The renormalization equations they derive have the differential form which is unusual in the real space renormalization theory.

This 2-dimensional renormalization scheme has a number of restrictions. So far it has not been possible to discuss the magnetic field exponents, the influence of other interactions than the nearest neighbor coupling, to set up a calculational scheme for the correlation functions or to treat random couplings. Therefore it seems worthwhile as an exploratory calculation to study these problems in a lower dimension $d = 1$, thus providing further insight in this new renormalization scheme, although all the results for the one-dimen-

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sional chain of this paper can be derived (often more easily) by other methods.

It turns out that for $d = 1$ an extra degree of freedom appears in the way a differential form of renormalization equations can be formulated. In section 2 we start out with this general approach and use this freedom to simplify the renormalization transformation in the case of spatially homogeneous hamiltonians (which is not possible for $d = 2$). The transformation we get applies to inhomogeneous hamiltonians as well. We also discuss the critical properties of the system for other allowed choices of the renormalization transformation.

In section 3 we calculate the free energy of a chain with arbitrary spatially inhomogeneous nearest-neighbor interactions. In section 4 we introduce the magnetic field and determine the flow pattern for the renormalization of systems with spatially homogeneous nearest-neighbor interactions in a magnetic field. We consider the chain with nearest neighbor and next nearest neighbor interactions in section 5.

In section 6 we show that the usual scheme to deal with correlation functions fails and we present a different method.

In section 7 we treat the random ferromagnet and the spin glass and we close the paper with a discussion of the results.

## 2. The transformation

In the renormalization group approach a map is constructed from a hamiltonian $\mathscr{H}(s)$ of $N$ spins to a new $\mathscr{H}'(s')$ with lesser, say $N'$, spins, such that $\mathscr{H}$ and $\mathscr{H}'$ lead to essentially the same free energy. In the paper of HSL[1]) this is achieved by the set of equations

$$e^{\mathscr{H}(s)} = \mathrm{Tr}'\, e^{\mathscr{H}_c(s',s)}, \tag{2.1a}$$

$$e^{\mathscr{H}(s')} = \mathrm{Tr}\, e^{\mathscr{H}_c(s',s)}. \tag{2.1b}$$

Tr(Tr') stands for a summation through all the $2^N(2^{N'})$ configurations of the $N(N')$ Ising spins. In our case the $N$ spins are located on a chain $\mathscr{C}$ and the $N' = N - 1$ new spins on a chain $\mathscr{C}'$ of the same length $L$ and thus with a larger lattice distance $a' \simeq a(1 + a/L)$ when $a$ is the lattice distance of $\mathscr{C}$ and $a/L$ is small (see fig. 1).

The coupling hamiltonian $\mathscr{H}_c(s', s)$ relates the spins of both systems and will be chosen as

$$\mathscr{H}_c(s', s) = \sum_X \left\{ p^+ \left( \frac{X + X_0}{2} \right) s(X_0) + p^- \left( \frac{X + X_0 + a}{2} \right) s(X_0 + a) \right\} s'(X). \tag{2.2}$$

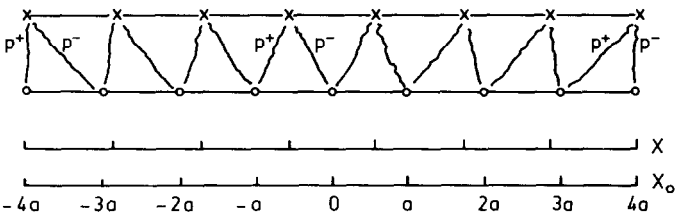


Fig. 1. The chains $\mathscr{C}$ (circles, coordinates $X_0$) and $\mathscr{C}'$ (crosses, coordinates $X$). $\mathscr{C}$ and $\mathscr{C}'$ are coupled by $p^+$ and $p^-$.

Here $X_0$ is a site of chain $\mathscr{C}$ and $X$ a site of $\mathscr{C}'$ related by (2.3), $s(X_0)$ the spin at site $X_0$ etc. and $p^\pm$ are coupling parameters which may vary in space and are given the coordinate corresponding to the middle of the two sites which they couple. The sum in (2.2) runs though all sites in the chain $\mathscr{C}'$, the points $X_0$ and $X$ coupled by $p^+$ being related by

$$\begin{cases} X_0 = X - (\frac{1}{2} + X/L)a \\ X = X_0 + (\frac{1}{2} + X_0/L)a' \end{cases} \tag{2.3}$$

ignoring terms of order $(a/L)^2$. The coordinate $X$ runs from $-L/2$ to $L/2$.

The form (2.2) is a straightforward specialization to $d = 1$ of the $\mathscr{H}_c(s', s)$ proposed by HSL. Here too we should think of the $p^\pm(X)$ as slowly varying in space, in a way to be specified later, and fulfilling the boundary conditions

$$p^+(-L/2) = \infty, \quad p^-(L/2) = \infty, \tag{2.4}$$

which express that the spins at the endpoints of $\mathscr{C}$ and $\mathscr{C}'$ are parallel. By taking $p^\pm$ positive, we confine ourselves to the ferromagnetic chain; the antiferromagnetic case will be discussed at the end of this section.

The structure of (2.1) is extremely simple for the choice (2.2) of $\mathscr{H}_c$ since the sums of new and old spins factorize. The results are expressions for $\mathscr{H}(s)$ and $\mathscr{H}'(s')$ of the form

$$\mathscr{H}(s) = \sum_{X_0} K(X_0 + a/2)s(X_0)s(X_0 + a) + G, \tag{2.5a}$$

$$\mathscr{H}'(s') = \sum_{X} K'(X + a/2)s'(X)s'(X + a) + G'. \tag{2.5b}$$

The $K$, $G$, $K'$ and $G'$ are functions of $p^\pm$ which can be presented conveniently by introducing the new variables

$$\begin{aligned} &\chi(x) = \operatorname{tgh} K(X), \quad \chi'(x) = \operatorname{tgh} K'(X), \\ &\varphi^+(x) = \operatorname{tgh} p^+(X), \quad \varphi^-(x) = \operatorname{tgh} p^-(X). \end{aligned} \tag{2.6}$$

Here we have introduced the scaled coordinate $x = X/L$. After some algebra we find from transformation (2.2), using (2.3)

$$\chi(x) = \varphi^+(x + (x - \tfrac{1}{2})a/2L)\varphi^-(x + (x + \tfrac{1}{2})a/2L), \tag{2.7a}$$

$$\chi'(x) = \varphi^+(x - (x - \tfrac{1}{2})a/2L)\varphi^-(x - (x + \tfrac{1}{2})a/2L). \tag{2.7b}$$

The expressions for $G$ and $G'$ are given in the next section. We are interested in the infinite chain, i.e. in the limit $a/L \to 0$. To zeroth and first order in $a/L$ we obtain from (2.7)

$$\chi(x) = \varphi^+(x)\varphi^-(x), \tag{2.8a}$$

$$\delta\chi(x) = \chi'(x) - \chi(x) = (a/L)\{\tfrac{1}{2}\varphi^-(x)\nabla\varphi^+(x) - \tfrac{1}{2}\varphi^+(x)\nabla\varphi^-(x) - x\nabla\chi(x)\}. \tag{2.8b}$$

Eq. (2.8a) yields for given $\chi(x)$ a restriction on the possible choices $\varphi^{\pm}(x)$, whereas (2.8b) leads to the flow equation for $\chi$ by putting $a/L = \delta t$ and writing $\chi(x, t)$ as the value of the interaction constant $\chi$ at 'time' $t$:

$$\chi(x, t) = \varphi^+(x, t)\varphi^-(x, t), \tag{2.9a}$$

$$\frac{\partial\chi(x, t)}{\partial t} = \tfrac{1}{2}\{\varphi^-(x, t)\nabla\varphi^+(x, t) - \varphi^+(x, t)\nabla\varphi^-(x, t)\} - x\nabla\chi(x, t). \tag{2.9b}$$

In view of (2.4) the $\varphi^{\pm}$ fulfil the boundary conditions

$$\varphi^+(-\tfrac{1}{2}, t) = 1, \quad \varphi^-(\tfrac{1}{2}, t) = 1. \tag{2.10}$$

Before we explore the full implication of (2.9) we investigate under what condition we can derive a renormalization equation for a spatially independent $\chi(x, t)$. As (2.9b) may be written as

$$\frac{\partial\chi(x, t)}{\partial t} = \frac{\chi(x, t)}{2}\{\nabla \ln(\varphi^+(x, t)/\varphi^-(x, t))\} - x\nabla\chi(x, t), \tag{2.11}$$

we see that spatial independence of $\chi$ requires $\ln \varphi^+/\varphi^-$ to be linear in $x$. Together with the boundary conditions (2.10) and (2.9a) we then obtain

$$\varphi^+(x, t) = [\chi(t)]^{x+1/2}, \tag{2.12a}$$

$$\varphi^-(x, t) = [\chi(t)]^{-x+1/2}, \tag{2.12b}$$

and the renormalization equation

$$\frac{\partial\chi(t)}{\partial t} = \chi(t)\ln(\chi(t)). \tag{2.13}$$

The solution of (2.13) is

$$\chi(t) = [\chi(0)]^{e^t}. \tag{2.14}$$

We will discuss (2.13) and (2.14) after generalizing the discussion to the chain with inhomogeneous interactions. Returning to this more general case we use the freedom of choice for $\varphi^{\pm}$ by writing

$$\frac{\varphi^{+}(x,t)}{\varphi^{-}(x,t)} = [\chi(x,t)]^{2\alpha(x)}. \tag{2.15}$$

We refrain from taking $\alpha(x)$ 'time' dependent, which means that $\alpha(x)$ remains the same along the renormalization trajectory, as we see no purpose for the freedom to allow for a 'time' dependence of $\alpha(x)$. From (2.9a) and (2.15), we find for $\varphi^{\pm}$

$$\varphi^{+}(x,t) = [\chi(x,t)]^{\alpha(x)+1/2}, \quad \varphi^{-}(x,t) = [\chi(x,t)]^{-\alpha(x)+1/2}. \tag{2.16}$$

In order that $\varphi^{\pm}$ fulfil boundary conditions (2.10) and to prevent $\varphi^{\pm}$ from exceeding 1, $\alpha(x)$ has to obey

$$\alpha(-\tfrac{1}{2}) = -\tfrac{1}{2}, \quad \alpha(\tfrac{1}{2}) = \tfrac{1}{2}, \tag{2.17a}$$

$$-\tfrac{1}{2} \leqslant \alpha(x) \leqslant \tfrac{1}{2}. \tag{2.17b}$$

The representation (2.16) has the advantage that $\varphi^{\pm} \to 1$ when $\chi \to 1$, as is required by (2.9a), and the fact that $\chi$, $\varphi^{\pm}$ are bounded by 1.

Inserting (2.15) into (2.11) and introducing $u(x,t) = -\ln \chi(x,t)$ we obtain

$$\frac{\partial u(x,t)}{\partial t} = u(x,t)\nabla\alpha(x) + (\alpha(x) - x)\nabla u(x,t). \tag{2.18}$$

The form (2.18) is particularly suited to discuss the behavior near the zero temperature fixed point $u^* = 0$ (or $\chi^* = 1$, $K^* = \infty$), as the equation is already linear in the deviation from the fixed point. Note that $u^* = 0$ is a fixed point of (2.18) independent of $\alpha(x)$.

One observes that for $\alpha(x) \equiv x$ we recover the generalization of (2.13) to the inhomogeneous case

$$\frac{\partial u(x,t)}{\partial t} = u(x,t), \tag{2.19a}$$

or equivalently

$$\frac{\partial \chi(x,t)}{\partial t} = \chi(x,t) \ln \chi(x,t). \tag{2.19b}$$

The eigenvalue equation for linear deviations $\zeta(x)$ from the fixed points $\chi^* = 1$ and $\chi^* = 0$ (the infinite temperature fixed point) has the form

$$y\zeta(x) = \zeta(x)[1 + \ln \chi^*] \tag{2.20}$$

and one finds that due to the local character of (2.19) (no gradient terms) the

eigenvalues are infinitely degenerate and that the eigenfunctions are local. In the infinite temperature fixed point $\chi^* = 0$, $y = -\infty$ and for the zero temperature fixed point $\chi^* = 1$, $y = 1$. This fixed point may be seen as the critical fixed point and leads to the correct[2]) correlation length exponent $\nu = y^{-1} = 1$.

For general $\alpha(x)$ the eigenvalue equation corresponding to (2.18) reads

$$yu(x) = u(x)\nabla\alpha(x) + [\alpha(x) - x]\nabla u(x), \tag{2.21}$$

$u(x)$ can be expressed in terms of $\alpha(x)$ as

$$u(x) = u(0) \exp \int_0^x \mathrm{d}x' \left[\frac{y - \nabla\alpha(x')}{\alpha(x') - x'}\right], \tag{2.22}$$

provided that $\alpha(x) \not\equiv x$. (The case $\alpha(x) \equiv x$ was treated previously). The eigenvalues $y$ are determined by the question whether $u(x)$ remains acceptable at the singular points of the integral which are located at the solutions of the equation $\alpha(x) = x$, which has at least two solutions at $x = \pm\frac{1}{2}$ on the basis of the boundary condition (2.17a).

We now discuss three typical curves for $\alpha(x)$ shown in fig. 2a.

i) For curve 1 we have $\nabla\alpha(-\frac{1}{2}) < 1 < \nabla\alpha(\frac{1}{2})$. Then
   - for $y > \nabla\alpha(\frac{1}{2})$ one finds $u(\frac{1}{2}) = 0$ and $u(-\frac{1}{2}) = \infty$ or $\chi(\frac{1}{2}) = 1$ and $\chi(-\frac{1}{2}) = 0$. Thus the curve of $\chi(x)$ behaves like curve 1 in fig. 2b,
   - for $\nabla\alpha(-\frac{1}{2}) < y < \nabla\alpha(\frac{1}{2})$ we find by the same considerations a curve of type 2 in fig. 2b for $\chi(x)$,
   - for $y < \nabla\alpha(-\frac{1}{2})$ the curve of type 3 in fig. 2b results for $\chi(x)$.

ii) For curve 2 of fig. 2a one has $\nabla\alpha(\frac{1}{2}) < 1 < \nabla\alpha(-\frac{1}{2})$ and the conclusions are
   - for $y > \nabla\alpha(-\frac{1}{2})$ curve 2 in fig. 2b for $\chi(x)$,
   - for $\nabla\alpha(\frac{1}{2}) < y < \nabla\alpha(-\frac{1}{2})$ curve 2 in fig. 2b for $\chi(x)$,
   - for $y < \nabla\alpha(\frac{1}{2})$ curve 1 in fig. 2b for $\chi(x)$.

In all these cases $u$ diverges in at least one point, thus $u(x)$ cannot be

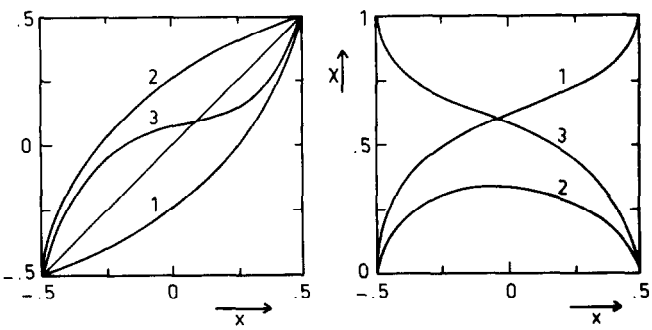


Fig. 2 a) Typical types of behavior for the function $\alpha(x)$ fulfilling conditions (2.17). b) Qualitative sketch of the eigenfunctions of the transformation when $\alpha(x) \not\equiv x$.

considered as a small deviation from the fixed point $u^* = 0$. In fact one sees from fig. 2b that at least at one of the endpoints the thermodynamic state is described by an infinite temperature rather than a zero temperature.

iii) When $\alpha(x)$ is of type 3 in fig. 2a we have an extra singularity in the integral (2.22) and the conclusion is again that $u(x)$ diverges at least at one point either at the intersection or at the boundary.

The conclusion is that no choice of $\alpha(x)$ other than $\alpha(x) \equiv x$ leads to a satisfactory eigenvalue picture.

Although we have confined ourselves so far to the ferromagnetic case ($K > 0$ and so $\chi > 0$), the antiferromagnetic chain ($K < 0, \chi < 0$) can be treated equally well: eqs. (2.9) show that for $\chi < 0$ one has to take one of the couplings, say $p^-(\varphi^-)$, negative. The analysis of this case can then be carried out analogously to the ferromagnetic case. Note that for $p^- < 0$ the boundary condition for $p^-$ reads $p^-(L/2) = -\infty$ (compare (2.4)), which expresses that the right endspins are antiparallel.

Finally we note a curious feature: since at every renormalization step the number of spins is reduced by 1 and $\delta t = a/L$, so that the number of spins drops exponentially with 'time', we have reduced the number of degrees of freedom by a factor of two at 'time' $\ln 2$. According to (2.14), for a homogeneous chain this corresponds to $\chi(\ln 2) = \chi^2(0)$. By performing a decimation, we reduce the number of degrees of freedom by the same factor, and find that also in that case $\chi$ transforms into $\chi^2$ (see ref. 2). Thus our transformation is the 'interpolation' of decimation.

## 3. The free energy

In this section we compute the free energy for an arbitrary inhomogeneous distribution of the nearest neighbor interaction along the lines given by HSL. However, in the $d = 2$ case one cannot calculate the free energy explicitly, as an explicit solution like (2.19) of the flow equation is lacking.

In order to prepare the ground for the free energy calculation we discuss the spin independent contribution $G$ and $G'$ to the hamiltonians in (2.5). Summing out a spin connected by a $p^+$ and $p^-$ bond to two other spins yields a spin independent term

$$g(p^+, p^-) = \ln 2 + \tfrac{1}{2} \ln(\cosh(p^+ + p^-) \cosh(p^+ - p^-)). \tag{3.1}$$

Thus the $G$ and $G'$ can be written as (using (2.3))

$$G = \sum_X g\left(p^+\left(X - \left(\frac{X}{L} + \tfrac{1}{2}\right)\frac{a}{2}\right), p^-\left(X - \left(\frac{X}{L} - \tfrac{1}{2}\right)\frac{a}{2}\right)\right), \tag{3.2a}$$

$$G' = \sum_{X_0} g\left(p^+\left(X_0 + \left(\frac{X_0}{L} + \tfrac{1}{2}\right)\frac{a}{2}\right), p^-\left(X_0 + \left(\frac{X_0}{L} - \tfrac{1}{2}\right)\frac{a}{2}\right)\right). \tag{3.2b}$$

These sums may be written as integrals up to errors involving the second derivative, which corresponds to terms of order $(a/L)^2$. Remembering that the density of points for the sums over $X$ and $X_0$ is $L/a' = (L/a)(1 - a/L)$ and $L/a$, respectively, we have after introduction of our scaled variable $x = X/L$,

$$G = \frac{L}{a}\left(1 - \frac{a}{L}\right)\int_{-1/2}^{1/2} dx g\left(\varphi^+\left(x - (x + \tfrac{1}{2})\frac{a}{2L}\right), \varphi^-\left(x - (x - \tfrac{1}{2})\frac{a}{2L}\right)\right), \tag{3.3a}$$

$$G' = \frac{L}{a}\int_{-1/2}^{1/2} dx g\left(\varphi^+\left(x + (x + \tfrac{1}{2})\frac{a}{2L}\right), \varphi^-\left(x + (x - \tfrac{1}{2})\frac{a}{2L}\right)\right), \tag{3.2b}$$

where $g(\varphi^+, \varphi^-)$ is the transcript of (3.1) in the $\varphi^\pm$ variables introduced in (2.6)

$$g(\varphi^+, \varphi) = \ln 2 + \tfrac{1}{2}\ln\left[\frac{1 - (\varphi^+\varphi^-)^2}{(1 - \varphi^{+2})(1 - \varphi^{-2})}\right]. \tag{3.4}$$

The free energy $Nf[K(X)]$ of the hamiltonian $\mathcal{H}(s)$ (2.5a) is defined as

$$e^{Nf[K(X)]+G} = \mathrm{Tr}\, e^{\mathcal{H}(s)}, \tag{3.5}$$

and similarly $N'f[K'(X)]$ of $\mathcal{H}'(s')$. From the definition (2.1) we have then the identity

$$Nf[K(X)] + G = N'f[K'(X)] + G'. \tag{3.6}$$

Comparing the leading orders yields the differential relation

$$\frac{\partial f[K(X, t)]}{\partial t} = f[K(X, t)] - C[K(X, t)], \tag{3.7}$$

where $C[K(X, t)]$ is the leading order of $G' - G$ of which the transcript in the $\chi, \varphi^\pm$ variables follows from (3.3) as

$$C[\chi(x, t)] = \int_{-1/2}^{1/2} dx c(\chi(x, t), x), \tag{3.8a}$$

with

$$c(\chi(x, t), x) = g(\varphi^+(x, t), \varphi^-(x, t)) + \frac{\partial g(\varphi^+(x, t), \varphi^-(x, t))}{\partial \varphi^+}\nabla\varphi^+(x, t)(x + \tfrac{1}{2}) + \frac{\partial g(\varphi^+(x, t), \varphi^-(x, t))}{\partial \varphi^-}\nabla\varphi^-(x, t)(x - \tfrac{1}{2}). \tag{3.8b}$$

According to (3.7) the free energy of a chain with a nearest neighbor interaction $K_0(x)$ (writing for convenience $K_0$ as a function of the scaled

coordinate) is given by

$$f[K_0(x)] = \int_0^\infty d\tau\, e^{-\tau} C[\chi(x,\tau)] = \int_0^\infty d\tau\, e^{-\tau} \int_{-1/2}^{1/2} dx c(\chi(x,\tau), x). \tag{3.9}$$

We will work out this expression with the aid of the transformation (2.19).

From direct computation, we know that the free energy will turn out to be

$$f[K_0(x)] = \int_{-1/2}^{1/2} dx \{\ln 2 \cosh K_0(x)\}. \tag{3.10}$$

In order to show the equivalence of (3.9) and (3.10) we would like to invert the order of the integrations in (3.9). However, this is not simply possible because $c(\chi(x), x)$ diverges at the boundaries, as is shown in fig. 3 for the case of a homogeneous interaction strength $K = 0.2$. To perform the 'time'-integral, we have to remove the boundary singularities first. By working out (3.8b) with the aid of (3.4) and (2.19) one finds that $c(\chi(x), x)$ may be written as

$$c(\chi(x), x) = \ln 2 + \tfrac{1}{2}\chi(x) \ln^2 \chi(x) \frac{d}{d\chi} \left\{ \frac{\ln[\chi(x)^{-1} - \chi(x)]}{\ln \chi(x)} \right\} + \nabla E(\chi(x), x), \tag{3.11}$$

where

$$E(\chi(x), x) = x \ln[\chi(x)^{-1} - \chi(x)] - \tfrac{1}{2}(x + \tfrac{1}{2}) \ln[\chi(x)^{-(x+1/2)} - \chi(x)^{x+1/2}] - \tfrac{1}{2}(x - \tfrac{1}{2}) \ln[\chi(x)^{x-1/2} - \chi(x)^{-(x-1/2)}]. \tag{3.12}$$

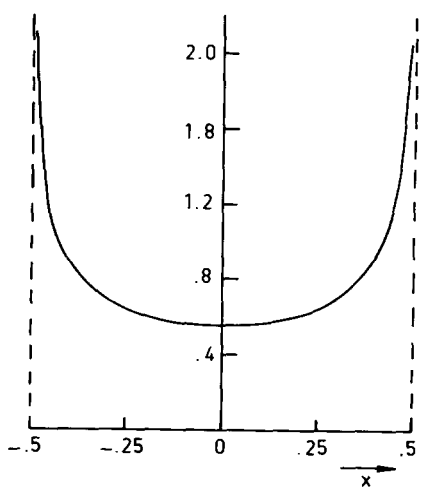


Fig. 3. The function $c[K, x]$ for $K = 0.2$. There are logarithmic singularities for $x = \pm\frac{1}{2}$.

Notice that the second term in the right hand side of (3.12) causes the divergence of $\nabla E$ at the left boundary, while the third term gives the divergence at the right boundary; however, in performing the $x$-integration over $c(\chi(x), x)$ in (3.11), the term $E$ drops out since $E$ vanishes for $x = \pm\frac{1}{2}$.

Substituting the remaining terms of (3.11) into (3.9), inverting the order of the integrations and using $\chi(x, t)$ from (2.19) as integration variable finally yields

$$f[K_0(x)] = \ln 2 + \int_{-1/2}^{1/2} \mathrm{d}x \, \frac{\ln \chi_0(x)}{2} \int_{\chi(x)=\chi_0(x)}^{\chi(x)=0} \mathrm{d}\left\{\frac{\ln(\chi(x)^{-1} - \chi(x))}{\ln \chi(x)}\right\}$$

$$= \ln 2 - \tfrac{1}{2} \int_{-1/2}^{1/2} \mathrm{d}x \, \ln(1 - \chi_0^2(x)) = \int_{-1/2}^{1/2} \mathrm{d}x \, \ln\{2 \cosh K_0(x)\}, \qquad (3.13)$$

in agreement with (3.10).

We remark that the starting formulae (3.8) and (3.9) suggest that only the free energy of the whole chain can be obtained, as $c$ is $x$-dependent even in the case of a spatially independent $\chi$. The expression (3.11) shows however that the explicit $x$-dependence of $c$ is contained in the divergence term $\nabla E$, with $E$ vanishing at the boundary. Leaving out $\nabla E$ in the expression for $c$ yields essentially a local relation for the free energy.

## 4. The Ising chain in a magnetic field

In this section we treat the case of a homogeneous Ising chain in a homogeneous magnetic field, i.e. a system with hamiltonian

$$\mathcal{H}(s) = \sum_{X_0} K s(X_0) s(X_0 + a) + \sum_{X_0} H s(X_0) + G. \qquad (4.1)$$

The extension to inhomogeneous systems can in principle be treated but leads to rather complicated (and unsolved) formulae.

The problem then is to find a $\mathcal{H}_c(s', s)$ representing $\mathcal{H}(s)$, given by (4.1), and yielding a renormalized $\mathcal{H}'(s')$ which differs only infinitesimally from $\mathcal{H}(s)$. We show that the following choice $\mathcal{H}_c(s', s)$ is satisfactory

$$\mathcal{H}_c(s', s) = \sum_{X_0} \left[ p^+ \left(\frac{X + X_0}{2}\right) s(X_0) + p^- \left(\frac{X + X_0 + a}{2}\right) s(X_0 + a) \right] s'(X)$$

$$+ \sum_{X_0} h(X_0) s(X_0) + \sum_{X} \tilde{h}(X) s(X), \qquad (4.2)$$

where $X_0$ and $X$ are again related by (2.3). For the difference $\mathcal{H}'(s') - \mathcal{H}(s)$ to be of the order of $a/L$, one has to impose the restriction that the difference

between $h$ and $\tilde{h}$ should at most be of order $a/L$. Therefore we define

$$\tilde{h}(X) = h(X) + v(X)a/L. \tag{4.3}$$

Since we will not discuss the free energy in this section, we leave the constant terms of the hamiltonian out of consideration. For $K$ and $H$ we find from (2.1a)

$$K = T\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0+a}{2}\right), \tilde{h}(X)\right), \tag{4.4a}$$

$$H = h(X_0) + U^+\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0+a}{2}\right), \tilde{h}(X)\right) + U^-\left(p^+\left(\frac{X+X_0-2a}{2}\right), p^-\left(\frac{X+X_0-a}{2}\right), \tilde{h}(X-a)\right), \tag{4.4b}$$

where

$$T(p^+, p^-, h) = \tfrac{1}{4}\ln\left[\frac{\cosh(p^+ + p^- + h)\cosh(-p^+ - p^- + h)}{\cosh(p^+ - p^- + h)\cosh(-p^+ + p^- + h)}\right], \tag{4.5a}$$

$$U^+(p^+, p^-, h) = \tfrac{1}{4}\ln\left[\frac{\cosh(p^+ + p^- + h)\cosh(p^+ - p^- + h)}{\cosh(-p^+ + p^- + h)\cosh(-p^+ - p^- + h)}\right], \tag{4.5b}$$

$$U^-(p^+, p^-, h) = U^+(p^-, p^+, h). \tag{4.5c}$$

According to (4.4b) there are two extra contributions to the new field on a spin of $\mathscr{C}$, one ($U^+$) from summing out the spin that is coupled by a $p^+$, the other ($U^-$) of the spin coupled by a $p^-$. We remark that since the left hand sides of eqs. (4.4a) and (4.4b) are independent of $X$, these equations in fact impose two conditions which $p^+$, $p^-$ and $h$ have to fulfil, reading

$$T_+\nabla p^+ + T_-\nabla p^- + T_h\nabla h = 0, \tag{4.6a}$$

$$(1 + U_h^+ + U_h^-)\nabla h + (U_+^+ + U_+^-)\nabla p^+ + (U_-^+ + U_-^-)\nabla p^- = 0, \tag{4.6b}$$

where we have used the abbreviations

$$T_+ = \frac{\partial T(p^+, p^-, h)}{\partial p^+}, \quad T_- = \frac{\partial T(p^+, p^- h)}{\partial p^-}, \quad \text{etc.} \tag{4.7}$$

For the interaction constants of $\mathscr{H}'(s')$ one finds from (2.1b)

$$K' = T\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0-a}{2}\right), h(X_0)\right), \tag{4.8a}$$

$$H' = \tilde{h}(X) + U^+\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0-a}{2}\right), h(X_0)\right) + U^-\left(p^+\left(\frac{X+X_0+2a}{2}\right), p^-\left(\frac{X+X_0+a}{2}\right), h(X_0+a)\right). \tag{4.8b}$$

By going over onto the scaled variables, using the relation between $X_0$ and $X$ and making a Taylor expansion in powers of $a/L$, we find from (4.4) and (4.8) the following set of transformation equations

$$K = T(p^+(x), p^-(x), h(x)), \tag{4.9a}$$

$$H = h(x) + U^+(p^+(x), p^-(x), h(x)) + U^-(p^+(x), p^-(x), h(x)), \tag{4.9b}$$

$$\frac{dK}{dt} = \tfrac{1}{2}T_+\nabla p^+ - \tfrac{1}{2}T_-\nabla p^- - T_h(x\nabla h + v), \tag{4.9c}$$

$$\frac{dH}{dt} = (1 - U_h^+ - U_h^-)(x\nabla h + v) + (U_h^- - U_h^+)\nabla h + \tfrac{1}{2}(3U_+^- - U_+^+)\nabla p^+ - \tfrac{1}{2}(3U_-^+ - U_-^-)\nabla p^-. \tag{4.9d}$$

At first sight the presence of a term in eqs. (4.9c) and (4.9d) containing $x$ explicitly, might seem somewhat strange since we are dealing with homogeneous interactions. It is caused by the fact that $v$ relates the interpolated fields $h$ and $\tilde{h}$ taken at the same place, while out of the centre the spins of $\mathscr{C}'$ are shifted with respect to the middle of the spins of $\mathscr{C}$.

As in the previous section, $p^+$ and $p^-$ are subject to boundary conditions (2.4). For the field-like terms $h$ and $v$ there are no a priori boundary conditions, since they do not couple the two chains directly.

A strategy to solve these equations could be to eliminate three of the variables, say $p^-$, $h$ and $v$, from eqs. (4.9). Then we are left with a differential equation for $p^+$. We can consider the solutions of this equation as functions of $dK/dt$ and $dH/dt$; then imposing the boundary conditions finally determines the values of $dK/dt$ and $dH/dt$. Instead of carrying out this program, we can use the boundary conditions immediately, since it turns out that $v$ plays no role at the boundaries. To show this, we consider the left boundary, where $p^+ \to \infty$; by Taylor expanding the functions $T$, $U^+$ and $U^-$ in powers of $e^{-2p^+}$, it is easily shown that $T_h$ and $(1 - U_h^+ - U_h^-)$, the coefficients of $(x\nabla h + v)$ in eqs. (4.9c) and (4.9d) respectively, become of the order of $e^{-2p^+}$. So, at the boundary $v$ does not influence the transformation provided $v$ does not diverge, and the leading terms in eqs. (4.9) only depend on $h$, $p^+$ and $p^-$. By eliminating them from these four equations, one finds that $dK/dt$ and $dH/dt$ are related by

$$\frac{dH}{dt} + 2\,\mathrm{tgh}\,H\,\frac{dK}{dt} = 0. \tag{4.10}$$

Although derived at the boundary, this relation will have to be fulfilled anywhere on the chain, since the interactions are homogeneous.

It is important to note that the sign of $p^-$ is immaterial in the derivation of (4.10); since a negative $p^-$ corresponds to $K < 0$, (see the discussion at the

end of section 2), (4.10) is valid for the ferromagnetic as well as the antiferromagnetic chain.

By introducing the variable $\psi = \text{tgh}\, H$, (4.10) can be written (as before, $\chi = \text{tgh}\, K$)

$$\frac{1}{\psi(1-\psi)(1+\psi)}\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{-2}{(1-\chi)(1+\chi)}\frac{\mathrm{d}\chi}{\mathrm{d}t}. \tag{4.11}$$

By eliminating the parameter $t$ and integrating we find that the flow equations obey

$$\psi(\chi) = \frac{(1-\chi)^2}{A(1+\chi)^2 + (1-\chi)^2}, \tag{4.12}$$

where $A$ is a constant to be determined from the initial conditions. The flow pattern given by (4.12) is shown in fig. 4; although (4.12) does not yield the fixed points and the direction of the flow, we expect all the corners in fig. 4 to be fixed points. Since the $\chi = 0$ region will be attractive (no ordering for finite values of $K$ and $H$), $\mathrm{d}\chi/\mathrm{d}t$ must vanish at the $\psi$-axis; then according to (4.11) $\mathrm{d}\psi/\mathrm{d}t$ vanishes too, which means that the $\psi$-axis is a fixed line. (A similar type of flow pattern was found by Nelson and Fisher[2]) for a discrete transformation).

To determine the magnetic exponent in the $\chi = 1$, $\psi = 0$ and $\chi = -1$, $\psi = 0$ fixed points, we assume that $\mathrm{d}\chi/\mathrm{d}t$ can be expanded in powers of $\psi$. However, because of symmetry between the $H > 0$ and $H < 0$ phases, $\mathrm{d}\chi/\mathrm{d}t$ cannot contain uneven powers in $\psi$. Consequently, we have $\mathrm{d}\chi/\mathrm{d}t = \chi \ln \chi + \mathcal{O}(\psi^2)$

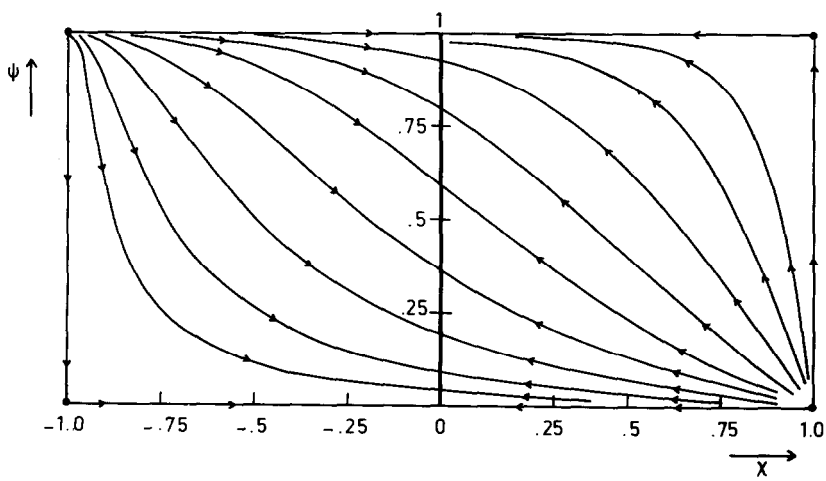


Fig. 4. The flowpattern according to (4.12). The points (1, 0), (1, 1), (−1,0) and (−1,1) are fixed points and the $\psi$-axis is a fixed line.

(where we have used (2.13)). Substituting this result in (4.11) yields

$$\frac{d\psi}{dt} = \psi + \mathcal{O}(\psi^2) \quad (\chi \to 1), \tag{4.13a}$$

$$\frac{d\psi}{dt} = -\psi + \mathcal{O}(\psi^2) \quad (\chi \to -1), \tag{4.13b}$$

corresponding with the relevant exponent $y_H = 1$ in the ferromagnetic fixed point and an irrelevant exponent ($y_H = -1$) in the antiferromagnetic fixed point. This was also found by Nelson and Fisher[2]).

As we see from fig. 4, the flow lines come out of the $\chi = -1$, $\psi = 1$ ($K = -\infty$, $H = \infty$) fixed point horizontally. This means that the expansion around the fixed point is of the form

$$\frac{d(1+\chi)}{dt} = B(1+\chi)(1-\psi) + \cdots, \tag{4.14a}$$

$$\frac{d(1-\psi)}{dt} = \frac{B}{2}(1-\psi)^2 + \cdots, \tag{4.14b}$$

where $B$ is some constant that can only be determined from the explicit solution of the transformation equations (4.9). Since there are no terms linear in the deviations from the fixed point, (4.14) shows that both the eigenvalues are marginal ($y_T = y_H = 0$); this expresses the competition between the antiparallel ordering (stimulated by $K \to -\infty$) and parallel ordering (stimulated by $H \to \infty$).

Finally, we derive a formal expression for the transformation equations. It was already stated that relation (4.10) must hold anywhere on the chain, since the interactions are homogeneous. This enables us to express $dK/dt$ in terms of $K$, $H$, $p^+$, $p^-$ and $h$ only by eliminating the combination $x\nabla h + v$ and $dH/dt$ from (4.9c) with the aid of (4.9d) and (4.10). One finds

$$\frac{dK}{dt} = F^+(p^+, p^-, h)\nabla p^+ + F^-(p^+, p^-, h)\nabla p^- + F^h(p^+, p^-, h)\nabla h, \tag{4.15}$$

where

$$F^+(p^+, p^-, h) = \tfrac{1}{2}\frac{T_+(1 - U_h^+ - U_h^-) + T_h(3U_+^- - U_+^+)}{1 - U_h^+ - U_h^- - 2\,\mathrm{tgh}\, H \cdot T_h}, \tag{4.16a}$$

$$F^-(p^+, p^-, h) = -\tfrac{1}{2}\frac{T_-(1 - U_h^+ - U_h^-) + T_h(3U_-^+ - U_-^-)}{1 - U_h^+ - U_h^- - 2\,\mathrm{tgh}\, H \cdot T_h}, \tag{4.16b}$$

$$F^h(p^+, p^-, h) = \frac{T_h(U_h^- - U_h^+)}{1 - U_h^+ - U_h^- - 2\,\mathrm{tgh}\, H \cdot T_h}. \tag{4.16c}$$

Because $dK/dt$ is independent of $x$, we can write this as

$$\frac{dK}{dt} = \int_{-1/2}^{1/2} dx[F^+(p^+, p^-, h)\nabla p^+ + F^-(p^+, p^-, h)\nabla p^- + F^h(p^+, p^-, h)\nabla h]. \tag{4.17}$$

Here, the first and second term of the integrand are equal, because $F^-(p^+, p^-, h) = -F^+(p^+, p^-, h)$ and $p^-(x) = p^+(-x)$ (in the case of a ferromagnetic chain). Moreover, $F^h(p^+, p^-, h)$ is an uneven function of $x$ and this allows us to write the transformation equation as

$$K = T(p^+, p^-, h), \tag{4.18a}$$

$$H = h + U^+(p^+, p^-, h) + U^-(p^+, p^-, h), \tag{4.18b}$$

$$\frac{dK}{dt} = 2\int_{\infty}^{K} dp^+ F^+(p^+, p^-, h) + 2\int_{H/2}^{h(0)} dh F^h(p^+, p^-, h), \tag{4.18c}$$

$$\frac{dH}{dt} = -2 \operatorname{tgh} H \frac{dK}{dt}. \tag{4.18d}$$

(we used the boundary condition $p^+(-\frac{1}{2}) = \infty$ which implies $p^+(\frac{1}{2}) = K$ and $h(-\frac{1}{2}) = H/2$); $h(0)$ follows from (4.18b) with $p^+(0) = p^-(0)$). In the first integration of (4.18c), $p^-$ and $h$ are supposed to be expressed in terms of $p^+$ by means of the first two equations, while similarly $p^+$ and $p^-$ are written in terms of $h$ in the second integrand.

We remark that in this formal expression $p^+$, $p^-$ and $h$ enter only as integration variables; one does not have to know their behavior on the chain to determine the transformation properties of $K$ and $H$.

## 5. Higher order interactions

To investigate the possibility of treating chains with higher order interactions, we will first consider the chain with next-nearest-neighbor (n.n.n.) interactions. In order that transformation (2.1) generates this kind of interactions in the hamiltonians $\mathcal{H}(s)$ and $\mathcal{H}'(s')$, another coupling parameter has to be introduced. This suggests the configuration of fig. 5, where each spin of the new chain $\mathcal{C}''$ is coupled with three spins of $\mathcal{C}$ by couplings $p^+$, $p^0$ and $p^-$. Note that because of symmetry $\mathcal{C}''$ has two spins less than $\mathcal{C}$. Therefore the relations between the coordinates $X_0$ and $X$ of spins of $\mathcal{C}$ and $\mathcal{C}''$ which are

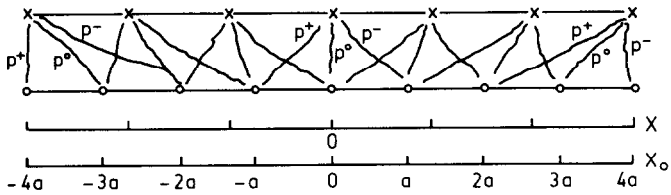


Fig. 5. The chains $\mathscr{C}$ (circles) and $\mathscr{C}''$ (crosses). $\mathscr{C}''$ has two spins less than $\mathscr{C}$ and is coupled to it by $p^+$, $p^-$ and $p^0$.

coupled by a $p^0$ coupling, now read

$$\begin{cases} X = X_0 + \dfrac{2aX_0}{L}, \\ X_0 = X - \dfrac{2aX}{L}. \end{cases} \tag{5.1}$$

Defining the coordinates of the coupling parameters in the same way as before, we can write the coupling hamiltonian as

$$\mathscr{H}_c(s', s) = \sum_X \left[ p^+ \left( \frac{X + X_0 - a}{2} \right) s(X_0 - a) + p^0 \left( \frac{X + X_0}{2} \right) s(X_0) + p^- \left( \frac{X + X_0 + a}{2} \right) s(X_0 + a) \right] s'(X). \tag{5.2}$$

A renormalization of a chain with homogeneous n.n. and n.n.n. interactions $K$ and $M$ respectively, would imply four constraints whereas there are only three degrees of freedom, $p^+$, $p^0$ and $p^-$. This is in general impossible and we are bound to suppose that at least one of the interactions is inhomogeneous. Since we do not want to make a specific choice in advance (either $K$ or $M$ homogeneous or none of them) we assume that both $K$ and $M$ are position-dependent. The transformation equations that can be derived from transformation (2.1) then read (as in the previous section, we leave the constant terms $G$ and $G'$ out of consideration and go over onto the scaled coordinate)

$$K(x) = V^+(p^+(x), p^0(x), p^-(x)) + V^-(p^+(x), p^0(x), p^-(x)), \tag{5.3a}$$

$$M(x) = V^0(p^+(x), p^0(x), p^-(x)), \tag{5.3b}$$

$$\frac{\partial K(x, t)}{\partial t} = V_+^+ \nabla p^+ - V_-^+ \nabla p^- + \tfrac{1}{2}(V_0^+ - V_0^-) \nabla p^0 - x \nabla K, \tag{5.3c}$$

$$\frac{\partial M(x, t)}{\partial t} = V_+^0 \nabla p^+ - V_-^0 \nabla p^- - x \nabla M. \tag{5.3d}$$

where now $\delta t = 2a/L$, since $\mathscr{C}''$ has two spins less than $\mathscr{C}$, and where

$$V^+(p^+, p^0, p^-) = \tfrac{1}{4} \ln\left[\frac{\cosh(p^+ + p^0 + p^-)\cosh(-p^+ + p^0 + p^-)}{\cosh(p^+ - p^0 + p^-)\cosh(p^+ + p^0 - p^-)}\right], \tag{5.4a}$$

$$V^0(p^+, p^0, p^-) = V^+(p^0, p^+, p^-), \tag{5.4b}$$

$$V^-(p^+, p^0, p^-) = V^+(p^-, p^0, p^+). \tag{5.4c}$$

In (5.3a) the n.n. interaction is the sum of two terms, since as fig. 5 shows, summing over one spin variable contributes to two n.n. interactions, except at the boundaries: the n.n. interaction at the left boundary of $\mathscr{C}$ is only given by $V^-$, while the one of $\mathscr{C}''$ is the sum of $V^+$ and $V^-$. Hence for the transformation to be infinitesimal, $V^+$ has to be zero at the left boundaries, while $V^-$ and $V^0$ are not. According to (5.4a)–(5.4c) this requires $p^+(-\frac{1}{2}) = \infty$ and similarly $p^-(\frac{1}{2}) = \infty$, the boundary conditions we used already in section 2. For the pure Ising case (only n.n. interactions), these boundary conditions cannot be derived by a similar consideration, since then $K$ and $K'$ always differ only infinitesimally if $p^+$ and $p^-$ are slowly varying. However, the reason why we took these boundary conditions in section 2 too is not only to have a close analogy with this case, but also because they are in agreement with the intuitive idea that spins lying close to each other are coupled strongly.

This relation between position on the chain and strength of the coupling does not seem to be valid in the centre of the chain, since for the configuration of fig. 5 this would suggest $p^0(0) = \infty$ implying $M(0) = 0$.

As in the case of only n.n. interactions (eq. (2.9)), eqs. (5.3) describe a group of transformations, depending on the specific choice we make. For instance, the term $x\nabla K$ drops out when $K$ is chosen homogeneous. A solution of the form (4.18) is now impossible, since always one of the two transformation equations will contain $x$ explicitly.

In confining ourselves to local hamiltonians of the form (5.2) one easily sees that the approach sketched above for n.n.n.-interactions cannot be generalized to arbitrary higher order interactions. By introducing a fourth coupling, a third neighbor interaction and a four-spin interaction would be generated. So we would have as many coupling parameters as interactions, and renormalization would in principle be possible. However, as soon as we introduce more coupling parameters we get more types of interaction couplings than parameters, and a closed set of renormalization equations cannot be obtained.

Though we have not been able to solve the transformation equations for a specific choice, a simple approximation is possible for small $p^0$ and homogeneous $M$. Expanding (5.3a) and (5.3b) in powers of $p^0$ yields

$$K(x) = \mathrm{tgh}(p^+(x) + p^-(x)) \cdot p^0(x) + \mathcal{O}(p^{03}), \tag{5.5a}$$

$$M = \tfrac{1}{2}\ln\left[\frac{\cosh(p^+(x)+p^-(x))}{\cosh(p^+(x)-p^-(x))}\right] + \mathcal{O}(p^{02}). \tag{5.5b}$$

Thus, for small $p^0$, $M$ is independent of $p^0$ while $K$ is of the order of $p^0$; moreover, eq. (5.5b) is exactly the same as the equivalence of (2.9a) for the n.n. interaction in section 2. This is not surprising as for $K = 0$ in this case, the two chains both decouple into two chains with a n.n. interaction of strength $M$. This problem is solved in section 2, and by introducing $\chi_M = \operatorname{tgh} M$, the solutions are (compare (2.12) and (2.13))

$$\frac{d\chi_M}{dt} = \chi_M \ln \chi_M + \mathcal{O}(p^{02}), \tag{5.6a}$$

$$\varphi^+(x) = \chi_M^{x+1/2} + \mathcal{O}(p^{02}), \tag{5.6b}$$

$$\varphi^-(x) = \chi_M^{-x+1/2} + \mathcal{O}(p^{02}), \tag{5.6c}$$

where, as before, $\varphi^+ = \operatorname{tgh} p^+$ and $\varphi^- = \operatorname{tgh} p^-$.
In this approximation of small $p^0$, which describes the behavior of two weakly coupled chains, eq. (5.5a) can be used to determine $p^0$ as a function of $K(x)$. Substituting the result in (5.3a) and using (5.6) yields

$$\frac{\partial K(x,t)}{\partial t} = \left[\frac{-2\chi_M \ln \chi_M}{(1+\chi_M)(1-\chi_M)} + \frac{(1+\chi_M)\ln\chi_M \operatorname{tgh}(x \ln \chi_M)}{2(1-\chi_M)}\right] K(x,t)$$
$$- \left[x + \frac{(1+\chi_M)\operatorname{tgh}(x\ln\chi_M)}{2(1-\chi_M)}\right] \nabla K(x,t) + \mathcal{O}(K^3). \tag{5.7}$$

According to (5.7), a homogeneous distribution will become inhomogeneous upon renormalization. However, in the neighborhood of the $K = 0$, $M = \infty$ ($\chi_M = 1$) fixed point, (5.7) simplifies to

$$\frac{\partial K(x,t)}{\partial t} = K(x,t), \tag{5.8}$$

corresponding to an exponent $y = 1$. This agrees with the prediction of Nelson and Fisher[2]), because they showed how the problem of the chain with n.n. and n.n.n. interactions can be translated into the one of the chain with n.n. interactions and magnetic field, implying the equivalence of this exponent and the magnetic exponent in the $K = \infty$, $H = 0$ fixed point of an Ising chain in a field.

## 6. The correlation function

In order to compute the spin–spin correlation function in this differential renormalization approach we can not use the usual formulae of the real space

renormalization method[3]) since in the derivation of those formulae the weight factor $P(s', s)$ is assumed to be independent of the hamiltonian, while the weight factor corresponding to (2.1) is

$$P(s', s) = \exp[\mathcal{H}_c(s', s) - \mathcal{H}(s)].$$

Owing to the explicit dependence of $P(s', s)$ on $\mathcal{H}(s)$, the method of ref. 3 cannot be applied.

In developing a somewhat different formalism, we take over the definition of constrained averages $\langle\ \rangle_s$ and $\langle\ \rangle_{s'}$

$$\langle A\rangle_s = \frac{\mathrm{Tr}'\, A\, \mathrm{e}^{\mathcal{H}_c(s',s)}}{\mathrm{Tr}'\, \mathrm{e}^{\mathcal{H}_c(s',s)}}, \quad \langle A\rangle_{s'} = \frac{\mathrm{Tr}\, A\, \mathrm{e}^{\mathcal{H}_c(s',s)}}{\mathrm{Tr}\, \mathrm{e}^{\mathcal{H}_c(s',s)}}, \tag{6.1}$$

while we denote by $\langle\ \rangle_{\mathcal{H}}$ and $\langle\ \rangle_{\mathcal{H}'}$ the usual averages with $\exp(\mathcal{H}(s))$ and $\exp(\mathcal{H}'(s'))$ as weight factor. For the constrained averages we have from (2.1) the identity

$$\langle\langle\ \rangle_s\rangle_{\mathcal{H}'} = \langle\langle\ \rangle_{s'}\rangle_{\mathcal{H}}, \tag{6.2}$$

which will form the basis for our derivation of the equations for the correlation function. Several combinations of spins on $\mathcal{C}$ and $\mathcal{C}'$ can be inserted in (6.2) leading to the desired result. The fastest way employs a product $s'(X)s(X_0 + la)$ of a spin on $\mathcal{C}'$ and one on $\mathcal{C}$ which is about $l$ lattice distances away. For the computation we need the basic ingredient

$$\langle s'(X)\rangle_s = Q^+\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0+a}{2}\right)\right) s(X_0) + Q^-\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0+a}{2}\right)\right) s(X_0+a), \tag{6.3a}$$

$$\langle s(X_0)\rangle_{s'} = Q^+\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0-a}{2}\right)\right) s'(X) + Q^-\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0-a}{2}\right)\right) s'(X-a), \tag{6.3b}$$

where

$$Q^+(p^+, p^-) = \tfrac{1}{2}(\mathrm{tgh}(p^+ + p^-) + \mathrm{tgh}(p^+ - p^-)), \tag{6.4a}$$

$$Q^-(p^+, p^-) = Q^+(p^-, p^+). \tag{6.4b}$$

Then, application of (6.2) to the form $s'(X)s(X_0 + a)$ yields

$$Q^+\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0+a}{2}\right)\right)\langle s(X_0)s(X_0+la)\rangle_{\mathscr{H}}$$
$$+ Q^-\left(p^+\left(\frac{X+X_0}{2}\right), p^-\left(\frac{X+X_0+a}{2}\right)\right)\langle s(X_0+a)s(X_0+la)\rangle_{\mathscr{H}}$$
$$= Q^+\left(p^+\left(\frac{X+X_0+2la}{2}\right), p^-\left(\frac{X+X_0+(2l-1)a}{2}\right)\right)\langle s'(X)s'(X+la)\rangle_{\mathscr{H}}$$
$$+ Q^-\left(p^+\left(\frac{X+X_0+2la}{2}\right), p^-\left(\frac{X+X_0+(2l-1)a}{2}\right)\right)$$
$$\times \langle s'(X)s'(X+(l-1)a)\rangle_{\mathscr{H}}. \quad (6.5)$$

The functional dependence on $K(X)$ of the two-point correlation function and its dependence on the coordinates of the two spins simplifies if we assume that $K(X)$ is a sufficiently smooth function. Indeed, if we assume to disregard derivatives of $K(X)$ higher than the first and indicate by $X_c$ the centre point between the two spins, one easily realizes that one can put simply

$$\left\langle s\left(X_c-\frac{la}{2}\right) s\left(X_c+\frac{la}{2}\right)\right\rangle_{\mathscr{H}} = g(l; K(X_c)), \quad (6.6)$$

where $g(l; K)$ is the correlation function of two spins at distance $l$ in a homogeneous system with interaction $K$. In (6.6), $X_c$ being the centre point, symmetry considerations allow us to exclude possible dependence on the gradient of $K(X)$. With this convention, (6.5) can be written in the form ($X_m$ is defined by $X_m = (X_0 + X + la)/2$)

$$Q^+\left(p^+\left(X_m-\frac{la}{2}\right), p^-\left(X_m-\frac{(l-1)a}{2}\right)\right) g\left[l; K\left(X_m-\left(\frac{X-X_0}{2}\right)\right)\right]$$
$$+ Q^-\left(p^+\left(X_m-\frac{la}{2}\right), p^-\left(X_m-\frac{(l-1)a}{2}\right)\right)$$
$$\times g\left[l-1; K\left(X_m-\left(\frac{X-X_0-a}{2}\right)\right)\right]$$
$$= Q^+\left(p^+\left(X_m+\frac{la}{2}\right), p^-\left(X_m+\frac{(l-1)a}{2}\right)\right) g\left[l; K\left(X_m+\left(\frac{X-X_0}{2}\right)\right)\right]$$
$$+ Q^-\left(p^+\left(X_m+\frac{la}{2}\right), p^-\left(X_m+\frac{(l-1)a}{2}\right)\right)$$
$$\times g\left[l-1; K\left(X_m+\left(\frac{X-X_0-a}{2}\right)\right)\right]. \quad (6.7)$$

If we now go over onto the scaled coordinates and use the variables $\chi$ and $\varphi^{\pm}$, by Taylor expanding, we find that the terms of order 1 drop out. Using the

fact that for $g$'s transcript in the $\chi$ variable

$$\frac{\partial g(l;\chi(x))}{\partial t} = \frac{\partial g(l;\chi(x))}{\partial \chi}\frac{\partial \chi(x)}{\partial t} = \chi(x)\ln\chi(x)\frac{\partial g(l;\chi(x))}{\partial \chi}, \tag{6.8a}$$

$$\nabla g(l;\chi(x)) = \frac{\partial g(l;\chi(x))}{\partial \chi}\nabla\chi(x), \tag{6.8b}$$

we find by comparing the terms of order $a/L$

$$\begin{aligned}&\left[Q^{+}\frac{\partial g(l;\chi)}{\partial\chi} + Q^{-}\frac{\partial g(l-1;\chi)}{\partial\chi}\right]\chi\ln\chi \\ &+\left[Q^{+}\frac{\partial g(l-1;\chi)}{\partial\chi}(x+\tfrac{1}{2}) + Q^{-}\frac{\partial g(l-1;\chi)}{\partial\chi}(x-\tfrac{1}{2})\right]\nabla\chi \\ &+[Q^{+}_{+}\nabla\varphi^{+}l + Q^{+}_{-}\nabla\varphi^{-}(l-1)]g(l;\chi) \\ &+[Q^{-}_{+}\nabla\varphi^{+}l + Q^{-}_{-}\nabla\varphi^{-}(l-1)]g(l-1;\chi) = 0,\end{aligned} \tag{6.9}$$

where we have omitted the spatial argument of the functions, since they are all to be evaluated at the same point. Furthermore we use the notation

$$Q^{+}_{+} = \frac{\partial Q^{+}}{\partial\varphi^{+}}, \quad Q^{+}_{-} = \frac{\partial Q^{+}}{\partial\varphi^{-}}, \quad \text{etc.} \tag{6.10}$$

Now, from (2.16) with $\alpha(x)\equiv x$, we have

$$\nabla\phi^{\pm} = \pm\varphi^{\pm}\left[\ln\chi + \frac{(x\pm\frac{1}{2})}{\chi}\nabla\chi\right]. \tag{6.11}$$

By substituting this into eq. (6.9), we get an equation of the form

$$D_1 + D_2\nabla\chi = 0, \tag{6.12}$$

where $D_1$ and $D_2$ are functions of $\chi, x, g(l;\chi)$ and $g(l-1;\chi)$. Since the correlation functions cannot depend on $\nabla\chi$, both $D_1$ and $D_2$ have to be zero, and we find two equations for the correlation functions, which after rearranging of some terms can be written as

$$\chi Q^{-}\frac{\partial g(l-1;\chi)}{\partial\chi} + Q^{+}_{-}\varphi^{-}(l-1)g(l;\chi) + Q^{-}_{-}\varphi^{-}(l-1)g(l-1;\chi) = 0, \tag{6.13a}$$

$$\chi Q^{+}\frac{\partial g(l;\chi)}{\partial\chi} + Q^{+}_{+}\varphi^{+}lg(l;\chi) + Q^{-}_{+}\varphi^{+}lg(l-1;\chi) = 0. \tag{6.13b}$$

By substituting $l\to l+1$ in (6.13a) and working out the functions we finally arrive at the set of equations

$$(1-\chi^2)\frac{\partial g(l;\chi)}{\partial\chi} + l[g(l+1;\chi) - g(l-1;\chi)] = 0, \tag{6.14a}$$

$$(1+\chi^2)g(l;\chi) - \chi[g(l+1;\chi) + g(l-1;\chi)] = 0. \tag{6.14b}$$

Note that $x$ does not appear explicitly any more. One convinces oneself easily that both eqs. (6.14) are satisfied by the solution

$$g(l;\chi) = [\chi]^l. \tag{6.15}$$

It is interesting to remark that each of the two equations is powerful enough to lead to the solution (6.15). Eq. (6.14a) can easily be converted into a partial differential equation for the Fourier transform $\tilde{g}$ of $g$, defined by

$$\tilde{g}(k;\chi) = \sum_l e^{-ikl} g(l;\chi). \tag{6.15}$$

Using this we obtain from (6.14a)

$$\left\{\tfrac{1}{2}(1-\chi^2)\frac{\partial}{\partial\chi} - \sin k \frac{\partial}{\partial k} - \cos k\right\} \tilde{g}(k;\chi) = 0. \tag{6.17}$$

For small $k$ (6.17) becomes

$$\left\{\tfrac{1}{2}(1-\chi^2)\frac{\partial}{\partial\chi} - k\frac{\partial}{\partial k} - 1\right\} \tilde{g}(k;\chi) = 0. \tag{6.18}$$

With simple dimensional considerations we can interpret (6.18) as expressing the fact that a change in $\tilde{g}$ due to a change in the natural unit of length of the theory, $a$, can be compensated by a suitable change in the coupling constant and a rescaling of $\tilde{g}$ itself.

We notice that (6.18) has the same formal structure of the equations one encounters for the correlation functions in the context of field theoretical renormalization group approaches.

## 7. The random Ising chain

In this section the coupling parameters in the hamiltonian (2.2) will be assumed independently random. As a consequence, the interactions in $\mathcal{H}(s)$ and $\mathcal{H}'(s')$ will also become independent random variables. This amounts to the realisation of a quenched disorder in the system.

Random Ising chains of this kind have been already studied by Grinstein et al.[4]), within the context of decimation transformations.

The probability distributions $W$ and $W'$ for the occurrence of interactions $K$ and $K'$ in the chains $\mathscr{C}$ and $\mathscr{C}'$ will be constructed from the probability distributions $W^+$ and $W^-$ in the combined system, where $W^+$ is the probability distribution for the parameter $p^+$ and $W^-$ for $p^-$. Again we prefer to work with the variables $\chi = \text{tgh}\, K$ and $\varphi^{\pm} = \text{tgh}\, p^{\pm}$. Now any combination of

$\varphi^+$ and $\varphi^-$ leads to a $\chi = \varphi^+\varphi^-$. Restricting ourselves to spatially homogeneous $W$ and $W'$ we may write in analogy of (2.7a)

$$W(\chi) = \int_{-1}^{1} d\varphi^+ \int_{-1}^{1} d\varphi^- W^+(\varphi^+, x + (x - \tfrac{1}{2})a/2L) \times W^-(\varphi^-, x + (x + \tfrac{1}{2})a/2L)\delta(\chi - \varphi^+\varphi^-), \tag{7.1a}$$

$$W'(\chi) = \int_{-1}^{1} d\varphi^+ \int_{-1}^{1} d\varphi^- W^+(\varphi^+, x - (x - \tfrac{1}{2})a/2L) \times W^-(\varphi^-, x - (x + \tfrac{1}{2})a/2L)\delta(\chi - \varphi^+\varphi^-). \tag{7.1b}$$

From (7.1a–b) differential flow equations will be derived below for $W$, provided it satisfies certain conditions. These equations, which fully describe the "time" evolution of the probability distribution under renormalization, can remarkably enough be reduced to the basic form (2.13), already encountered in the study of pure systems.

Without loss of generality we can write the probability distributions $W$ and $W'$ in the form

$$W(\chi) = \rho\delta(\chi) + W_>(\chi) + W_<(\chi), \tag{7.2a}$$

$$W'(\chi) = \rho'\delta(\chi) + W'_>(\chi) + W'_<(\chi), \tag{7.2b}$$

where the functions with $>$ and $<$ subscripts are identically zero for $\chi < 0$ and $\chi > 0$, respectively, and satisfy conditions of the type

$$\lim_{\epsilon \to 0^+} \int_0^{\epsilon} W_>(\chi)\, d\chi = 0. \tag{7.3}$$

The $W^+$ and $W^-$ probability distributions will also be put in a form analogous to (7.2a–b), namely

$$W^{\pm}(\varphi^{\pm}, x) = \rho^{\pm}(x)\delta(\varphi^{\pm}) + W^{\pm}_{\gtrless}(\varphi^{\pm}, x) + W^{\pm}_{\lessgtr}(\varphi^{\pm}, x), \tag{7.4}$$

with the same meaning as above of $>$ and $<$ subscripts.

This rewriting of the probability distribution functions is expedient in order to put the set (7.1a–b) in algebraic form via the introduction of Laplace transforms.

The last task is easily accomplished by expressing in terms of the new variables

$$\begin{aligned} u &= -\ln \chi, \\ v^{\pm} &= -\ln \varphi^{\pm}, \end{aligned} \qquad \chi, \varphi^{\pm} \geqslant 0, \tag{7.5a}$$

all the above functions with $>$ subscripts, and in terms of

$$\begin{aligned} u &= -\ln|\chi|, \\ v^{\pm} &= -\ln|\varphi^{\pm}|, \end{aligned} \qquad \chi, \varphi^{\pm} \leqslant 0, \tag{7.5b}$$

those with $<$ subscripts. Thus, in these new variables, both $>$ and $<$ functions are defined and eventually different from 0 on the positive semi-axis.

Introducing Laplace transforms for all these functions, hereafter denoted by $w$ characters, (7.1a–b) can be finally converted into

$$\begin{cases} \rho = \rho^{+}\left(x + (x-\frac{1}{2})\dfrac{a}{2L}\right) + \rho^{-}\left(x + (x+\frac{1}{2})\dfrac{a}{2L}\right) \\ \qquad - \rho^{+}\left(x + (x-\frac{1}{2})\dfrac{a}{2L}\right)\rho^{-}\left(x + (x+\frac{1}{2})\dfrac{a}{2L}\right), \\ \rho' = \rho^{+}\left(x - (x-\frac{1}{2})\dfrac{a}{2L}\right) + \rho^{-}\left(x - (x+\frac{1}{2})\dfrac{a}{2L}\right) \\ \qquad - \rho^{+}\left(x - (x-\frac{1}{2})\dfrac{a}{2L}\right)\rho^{-}\left(x - (x+\frac{1}{2})\dfrac{a}{2L}\right), \end{cases} \tag{7.6a}$$

$$\begin{cases} w_{>}(z) = w_{>}^{+}\left(z, x + (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{>}^{-}\left(z, x + (x+\frac{1}{2})\dfrac{a}{2L}\right) \\ \qquad + w_{<}^{+}\left(z, x + (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{<}^{-}\left(z, x + (x+\frac{1}{2})\dfrac{a}{2L}\right), \\ w'_{>}(z) = w_{>}^{+}\left(z, x - (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{>}^{-}\left(z, x - (x+\frac{1}{2})\dfrac{a}{2L}\right) \\ \qquad + w_{<}^{+}\left(z, x - (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{<}^{-}\left(z, x - (x+\frac{1}{2})\dfrac{a}{2L}\right), \\ w_{<}(z) = w_{>}^{+}\left(z, x + (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{<}^{-}\left(z, x + (x+\frac{1}{2})\dfrac{a}{2L}\right) \\ \qquad + w_{<}^{+}\left(z, x + (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{>}^{-}\left(z, x + (x+\frac{1}{2})\dfrac{a}{2L}\right), \\ w'_{<}(z) = w_{>}^{+}\left(z, x - (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{<}^{-}\left(z, x - (x+\frac{1}{2})\dfrac{a}{2L}\right) \\ \qquad + w_{<}^{+}\left(z, x - (x-\frac{1}{2})\dfrac{a}{2L}\right) w_{>}^{-}\left(z, x - (x+\frac{1}{2})\dfrac{a}{2L}\right), \end{cases} \tag{7.6b}$$

where, for example

$$w_{>}(z) = \int_0^{\infty} \mathrm{d}u\, \mathrm{e}^{-zu} W_{>}(\chi(u))\, \mathrm{d}\chi(u)/\mathrm{d}u, \tag{7.7}$$

$\chi$ being given by (7.5a) in terms of $u$.

In the following we will always consider only real $z$ arguments in the Laplace transforms.

As far as (7.6a) is concerned, we notice that it is not independent of (7.6b). Indeed, for all distributions the $\rho$-parameters are connected with the $z = 0$ components of the $>$ and $<$ transforms, as a consequence of normalization of total probabilities. The relations are

$$\begin{cases} \rho + w_>(0) + w_<(0) = 1, \\ \rho^{\pm}(x) + w^{\pm}_>(0, x) + w^{\pm}_<(0, x) = 1. \end{cases} \tag{7.8}$$

Thus, according to (7.8), in the following we will not care about deriving the "time" evolution of $\rho$: this will follow automatically, once given those of $w_>$ and $w_<$, which are obtainable from (7.6b).

In deriving the last ones, let us first consider the simplified situation in which all the probability distributions (7.2) and (7.4) are different from zero, say, only for positive values of the variables. This amounts to put equal to zero all $<$ transforms in (7.6b); so we are left with

$$\begin{cases} w_>(z) = w^+_>\left(z, x + (x - \tfrac{1}{2})\frac{a}{2L}\right) w^-_>\left(z, x + (x + \tfrac{1}{2})\frac{a}{2L}\right), \\ w'_> = w^+_>\left(z, x - (x - \tfrac{1}{2})\frac{a}{2L}\right) w^-_>\left(z, x - (x + \tfrac{1}{2})\frac{a}{2L}\right), \end{cases} \tag{7.9}$$

which has exactly the same structure as (2.7a–b). The formal identity with the pure system treatment is extended also to the boundary conditions, which in the present case, must be fixed as

$$w^+_>(z, -\tfrac{1}{2}) \equiv 1, \quad w^-_>(z, +\tfrac{1}{2}) \equiv 1, \tag{7.10a}$$

or, in a more transparent form,

$$W^+(\varphi^+, -\tfrac{1}{2}) = \delta(\varphi^+ - 1), \quad W^-(\varphi^-, +\tfrac{1}{2}) = \delta(\varphi^- - 1). \tag{7.10b}$$

Those above are a natural generalization of the condition of the conditions adopted in section 2. Indeed, they guarantee a complete parallelism of the spins at the boundaries of $\mathscr{C}$ and $\mathscr{C}'$, like (2.10) in the pure system case.

Eq. (7.9) and (7.10a–b) finally lead to the differential flow equation

$$\frac{\partial w_>(z, t)}{\partial t} = w_>(z, t) \ln w_>(z, t), \tag{7.11}$$

the possible fixed points being $w^*_>(z) \equiv 0$ ($\rho^* = 1$) and $w^*_>(z) \equiv 1$ ($\rho^* = 0$). The former one corresponds to $W^*(\chi) = \delta(\chi)$, whereas the latter implies $W^*(\chi) = \delta(\chi - 1)$, and is the pure ferromagnetic zero-temperature fixed point.

To get a qualitative picture of the flow pattern corresponding to (7.11), we can, for example, particularize it to the case of a two parameter distribution

of the form

$$W(\chi) = \rho\delta(\chi) + (1-\rho)\delta(\chi - \chi_0), \quad \chi_0 > 0, \tag{7.12}$$

which represents the most simple situation of dilution in an Ising ferromagnet. Substituting (7.12) in (7.11) we easily find

$$\frac{\mathrm{d}}{\mathrm{d}t}[1-\rho(t)] = [1-\rho(t)]\ln[1-\rho(t)], \quad \frac{\mathrm{d}\chi_0(t)}{\mathrm{d}t} = \chi_0(t)\ln\chi_0(t), \tag{7.13}$$

whose flow lines in the plane $(\chi_0, 1-\rho)$ are schematically reported in fig. 6a.

We notice that the problem of the dilute Ising ferromagnet, in the $T \to 0$ ($\chi_0 \to 1$) limit, becomes equivalent to the bond percolation problem, with bond occupation probability equal to $(1-\rho)$ [5]. So the renormalization equation, derived for $1-\rho$ above, is also appropriate for the probability in 1-dimensional bond percolation.

We can now consider the more general case of probability distributions not restricted to positive values. To this purpose it is useful to introduce symmetric ($S$) and anti-symmetric ($A$) quantitites, defined, in terms of $>$ and $<$ transforms, according to the prescription symbolically given below

$$S(A) = > + (-) <. \tag{7.14}$$

In the $S$- and $A$-quantities (7.6b) decouples completely, giving rise to the sets

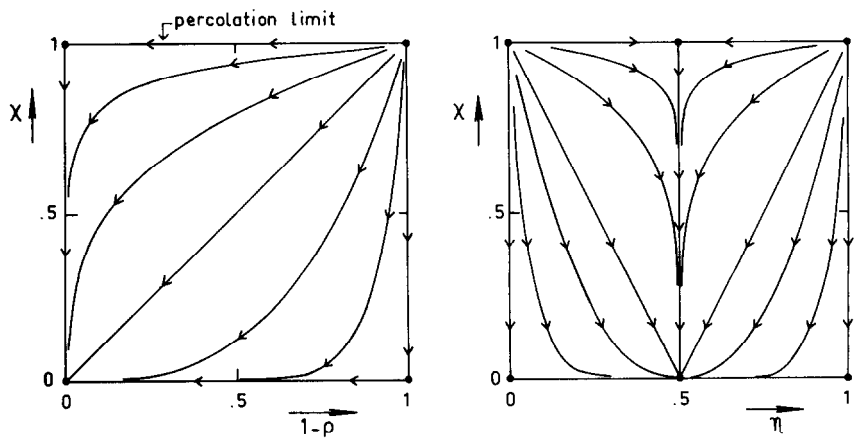


Fig. 6 a) The flow pattern according to (7.13) for the dilute ferromagnet. The line $\chi = 1$ corresponds to the percolation limit. b) The flow pattern according to (7.20) for a spin-glass distribution. The point (0.5, 1) is the zero temperature spin glass fixed point.

of equations

$$\begin{cases} w_S(z) = w_S^+\left(z, x + (x - \frac{1}{2})\frac{a}{2L}\right) w_S^-\left(z, x + (x + \frac{1}{2})\frac{a}{2L}\right), \\ w'_S(z) = w_S^+\left(z, x - (x - \frac{1}{2})\frac{a}{2L}\right) w_S^-\left(z, x - (x + \frac{1}{2})\frac{a}{2L}\right), \end{cases} \tag{7.15a}$$

$$\begin{cases} w_A(z) = w_A^+\left(z, x + (x - \frac{1}{2})\frac{a}{2L}\right) w_A^-\left(z, x + (x + \frac{1}{2})\frac{a}{2L}\right), \\ w'_A(z) = w_A^+\left(z, x - (x - \frac{1}{2})\frac{a}{2L}\right) w_A^-\left(z, x - (x - \frac{1}{2})\frac{a}{2L}\right). \end{cases} \tag{7.15b}$$

To treat (7.15a–b) in full generality, however, we now encounter a more complex situation in cases, in which $w_A(z)$ takes on both positive and negative values. Before discussing in more detail these problems, we first restrict ourselves to $W$ functions corresponding to either $w_A(z) \geq 0$ or $w_A(z) \leq 0$ for all $z$.

The first case, for example, corresponds to a situation in which ferromagnetic interactions decisively predominate over antiferromagnetic ones. As can be seen immediately from (7.15b), it is possible to reproduce a positive definite $w_A(z)$ with $w_A^+(z, x)$ and $w_A^-(z, x)$ both $\geq 0$ for all $z$. Furthermore one can convince oneself that the appropriate boundary conditions are

$$\begin{aligned} &w_S^+(z, -\tfrac{1}{2}) \equiv 1, \quad w_S^-(z, +\tfrac{1}{2}) \equiv 1, \\ &w_A^+(z, -\tfrac{1}{2}) \equiv 1, \quad w_A^-(z, +\tfrac{1}{2}) \equiv 1. \end{aligned} \tag{7.16}$$

These are equivalent to (7.10a–b) and thus consistent with a predominantly ferromagnetic situation. It is also evident that (7.16) and (7.15a–b) lead to equations of the form (7.11) for both $w_S$ and $w_A$, in this case. The conditions to be imposed when $w_A(z) \leq 0$ for all $z$, will be

$$\begin{aligned} &w_S^+(z, -\tfrac{1}{2}) \equiv 1, \quad w_S^-(z, +\tfrac{1}{2}) \equiv 1, \\ &w_A^+(z, -\tfrac{1}{2}) \equiv 1, \quad w_A^-(z, +\tfrac{1}{2}) \equiv -1, \end{aligned} \tag{7.17}$$

where now we have assumed the quantity $w_A^-(z, x)$ to be negative for all $z$ and $x$.

The conditions (7.17) are consistent with $w_>^+(z, -\frac{1}{2}) \equiv 1$ and $w_<^-(z, +\frac{1}{2}) \equiv 1$, the latter identity leading now to a complete antiparallelism of the spins at the right borders of $\mathscr{C}$ and $\mathscr{C}'$, as we expect it should be for antiferromagnetic situations.

The resulting equation for $w_A$ now becomes

$$\frac{\partial w_A(z, t)}{\partial t} \equiv w_A(z, t) \ln[-w_A(z, t)]. \tag{7.18}$$

whereas $w_S$ behaves as in the previous case. The equations for $w_S$ and $w_A$ in the two cases considered above lead to the expected zero temperature spin glass fixed point, with the correct stability properties[4]). This fixed point is given by $w_S^*(z) = 1$, $w_A^*(z) = 0$, which means $W^*(\chi) = \frac{1}{2}\delta(\chi - 1) + \frac{1}{2}\delta(\chi + 1)$. Also in this case it is interesting to study in detail the flows of particularly simple distributions, like that given by

$$W(\chi) = \eta\delta(\chi - \chi_0) + (1 - \eta)\delta(\chi + \chi_0), \quad 0 \leqslant \eta \leqslant 1, \quad 0 \leqslant \chi_0 \leqslant 1 \tag{7.19}$$

which is a typical Ising spin-glass distribution. For it one has $w_A(z) \gtrless 0$, according to whether $\eta \gtrless \frac{1}{2}$. We obtain very easily the following flow equations for $\eta$ and $\chi_0$:

$$\frac{d\eta}{dt} = \begin{cases} (\eta - \frac{1}{2}) \ln(2\eta - 1), & \eta > \frac{1}{2} \\ (n - \frac{1}{2}) \ln(1 - 2\eta), & \eta < \frac{1}{2} \end{cases} \tag{7.20}$$

$$\frac{d\chi_0}{dt} = \chi_0 \ln \chi_0,$$

whose corresponding flow lines are sketched in fig. 6b.

In the more general case that $w_A(z)$ changes sign at $z_0$, say

$$\begin{aligned} &w_A(z) > 0 \quad \text{for } z < z_0, \\ &w_A(z) < 0 \quad \text{for } z > z_0, \end{aligned} \tag{7.21}$$

the equations (7.15) are still valid. Although we would be tempted to derive the flow equations in the same way as before, this is impossible. To see this, notice that because of symmetry in the middle of the chains one must have $|w_A^+(z, 0)| = |w_A^-(z, 0)|$; then, from the terms of order 1 in (7.15b) we have

$$|w_A^+(z, 0)| = |w_A^-(z, 0)| = |w_A(z)|^{1/2}. \tag{7.22}$$

Now when $w_A(z)$ vanishes linearly at $z_0$, (7.22) shows that $w_A^\pm$ will become singular at $z_0$. This however is not allowed since according to (7.14) and (7.7) the $w_A^\pm$ are defined as the difference of the Laplace transforms (for real $z$) of positive definite and normalized functions. By their definition, the $w^\pm$ can never become singular for positive real $z$, and so we have no acceptable $w^\pm$ in the case (7.21).

In order to understand this somewhat embarrassing situation it is illuminating to make a comparison with a decimation transformation, that can always be constructed. The variables we used are also appropriate to describe this kind of transformation. As an example, notice that upon summing out every two out of three spins we get the renormalization transformation

$$w'_S(z) = w_S(z)^3, \tag{7.23a}$$

$$w'_A(z) = w_A(z)^3. \tag{7.23b}$$

We remark that distributions for which $w_A(z)$ is either $>0$ or $<0$ for all $z$ have fixed points $w_A(z) = \pm 1$ (pure ferromagnetic and antiferromagnetic fixed points) and $w_A(z) = 0$ (spin glass fixed point). Although the flow of distributions of type (7.21) is well defined and has a fixed point $w_A(z) = 1$ for $z < z_0$ and $w_A(z) = -1$ for $z > z_0$, this fixed point lies outside the space of allowed functions, since a discontinuous Laplace transform is not allowed. This anomalous behavior is of the same type as the one we encounter with our infinitesimal transformation.

## 8. Discussion

We have analyzed the possible renormalization treatments of Ising spin systems on a linear chain and come to the conclusion that the form (2.19) is the only acceptable one in view of the zero temperature fixed point structure. This form can be characterized as interpolating the decimation transformations. The same form is encountered in quenched random systems as well as the percolation problem although the physical interpretation of the variable is quite different (the Laplace transform of the probability distribution and the bond occupation probability, respectively).

The method of derivation hopefully provides some clues for so far unresolved problems in higher dimensions; of the possible extensions of our analysis we would like to mention the following.

1) Other coupling constants in the coupled system, such as a magnetic field, which in contrast to a pair interaction are not symmetric in the two systems have to be introduced in pairs (e.g. $h(X_0)$ and $\tilde{h}(X)$) which may be infinitesimally different. The apparant arbitrariness in the renormalization equations due to this infinitesimal difference is then removed again by the boundary conditions. A similar ambiguity appears in the two-dimensional case when one tries to introduce the magnetic field.

2) The differential renormalization procedure employs in general spatially dependent interaction constants not necessarily for the original and renormalized systems but anyway for the coupled system Hamiltonian $\mathcal{H}_c(s', s)$ which induces the transformation. In this context it seems that only the free energy of the whole system can be discussed although the system may be viewed as a set of locally thermodynamical systems. In two dimensions the solutions for the free energy requires so far a complete solution for the renormalization trajectory. In one dimension we managed to recast the formulae in such a way that the free energy can be obtained locally.

3) Exploration of the spatially inhomogeneous case leads to a set of two relations for the correlation function each of which is powerful enough to determine the correlation function. The same technique can be used in two dimensions. When applied to the nearest neighbor correlation it yields also here sufficient information to determine the nearest neighbor correlation function.

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