

Force Response as a Probe of the Jamming Transition

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ABSTRACT: A few years ago a “jamming diagram” has been proposed as a nonequilibrium phase diagram to describe the physics of glass transitions and granular materials in a unified way. Granular systems “jam”, i.e. develop a yield stress, when their packing fraction reaches a certain critical value upon compression. At zero temperature and load, this transition from “liquid” to “disordered solid” is governed by a critical point. We calculate the linear response to a point force for frictionless packings of various pressure. From the averaged response we obtain the effective Young modulus and Poisson ratio of the packings through fitting the coarse grained behaviour to continuum elastic theory. The elastic coefficients obtained this way show scaling behaviour near the jamming point.

1 INTRODUCTION

Many systems, including granular packings, glassy materials, and foams undergo a transition from a floppy, free flowing phase to a solid-like elastic phase as the appropriate parameters of the system are varied. The proposed interpretation in terms of a so-called “jamming transition” (Liu and Nagel 1998) has received a lot of attention because of its potential to be a common framework to describe particle systems with short range interactions of widely varying physical origin. The practical definition of the jammed state is that the internal stresses in the system are not relaxed to zero during experimentally available time scales. Dry granular media, with purely repulsive intergrain interactions, present ideal systems to study the jamming transition.

To fix the ideas, consider a system of macroscopic Hertzian spheres confined in a box so large that they fit in without touching or overlapping. Now quasistatically decrease the volume of the box increasing the packing fraction ϕ . At some point the grains cannot avoid touching each other anymore, and start to build up a pressure; the systems jams. For particles with finite-range repulsive interactions, the density at which this happens is sharply defined (O’Hern *et al.* 2003). Hence, for simple systems with zero shear Σ and zero temperature T , there is a sharply defined jamming transition, indicated by point J in the jamming diagram (Figure 1).

O’Hern *et al.* (O’Hern *et al.* 2003) have shown that

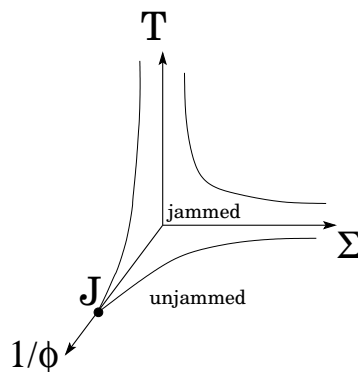


Figure 1: The Jamming phase diagram. The axes are inverse density $1/\phi$, temperature T , and shear load Σ . Point J is the well-defined transition at zero temperature and shear. [after (O’Hern *et al.* 2003)]

the granular system just beyond this point exhibits scaling behaviour as a function of pressure or density. This is reminiscent of critical phase transitions, and these findings immediately raise the question to what extent the jamming transition at $T = \Sigma = 0$ is similar to a second order phase transition. In particular, can we identify a length scale that diverges as the transition is approached?

In this paper we investigate the behaviour of a granular system as $p \rightarrow 0$, approaching the jamming transition from the jammed (solid-like) side. In our system, the transition is induced by changing the degree of confinement. While in everyday life most granular materials are confined by gravity, we confine, un-

der zero gravity, our granular system in a box with a known force acting on the sides of the box, to avoid depth dependent pressure. We have studied the critical behaviour as p is lowered from various angles, by analysing elastic properties and vibrational spectra, but focus here on the scaling behaviour of the effective elastic properties obtained from the system's response to a point loading.

2 RESPONSE TO EXTERNAL LOADS

In a jammed granular system (granular solid) at rest, there is a network of contacts between grains. Each contact carries a force such that there is force balance on each particle. If an external force is applied on the system, the contact forces change in a way which is governed by the contact force law and the condition that force balance on each particle is restored. We consider the linear regime, where the applied force is much smaller than the equilibrium forces, and breaking of contacts is negligible. The model system consists of slightly polydisperse spheres with a Hertzian contact law: $f \propto d^{3/2}$, where f is the force and d is the overlap length of the contact.

The numerical procedure consists of two main stages: a molecular dynamics part where a packing of $N = 10,000$ particles is generated, and a response calculation part which involves only linear algebra and no more dynamics.

2.1 Packing generation

The two-dimensional packing are generated by compressing a dissipative granular gas, for details see (Somfai *et al.* 2004). The packings are periodic in the x -direction, and are confined between hard walls in the y -direction. They are made with pressures ranging from $p = 10^{-2}$ to $p = 10^{-6}$, in units of the Young modulus of the grains' material.

2.2 Response calculating procedure

The linear response is calculated using what in condensed matter physics is called the *dynamical matrix*. It contains all information about the geometry of the contact network and the stiffnesses of the contacts. For the nonlinear force law used here, these stiffnesses depend on the equilibrium forces before any external load is applied. More precisely, it is the $2N \times 2N$ matrix of second derivatives of the elastic energy of the system with respect to the coordinates of the particles. It relates the displacements of all particles in the packing in response to the external forces acting on all particles in the packing as $MU = F_{\text{ext}}$, where M denotes the dynamical matrix, $U = (u_x^{(1)}, u_y^{(1)}, u_x^{(2)}, u_y^{(2)}, \dots, u_y^{(N)})$ is a vector collecting the displacements of all particles, and $F_{\text{ext}} = (F_x^{(1)}, F_y^{(1)}, F_x^{(2)}, F_y^{(2)}, \dots, F_y^{(N)})$ contains the exter-

nal forces on all particles.

For a stable packing, the matrix M is symmetric positive-definite, so it can be inverted to find the displacement field U for a given external load F_{ext} :

$$U = M^{-1}F_{\text{ext}}, \quad (1)$$

which we do numerically using a conjugate gradient algorithm. The same procedure has recently been applied to packings of Lennard-Jones particles (Leonforte *et al.* 2004).

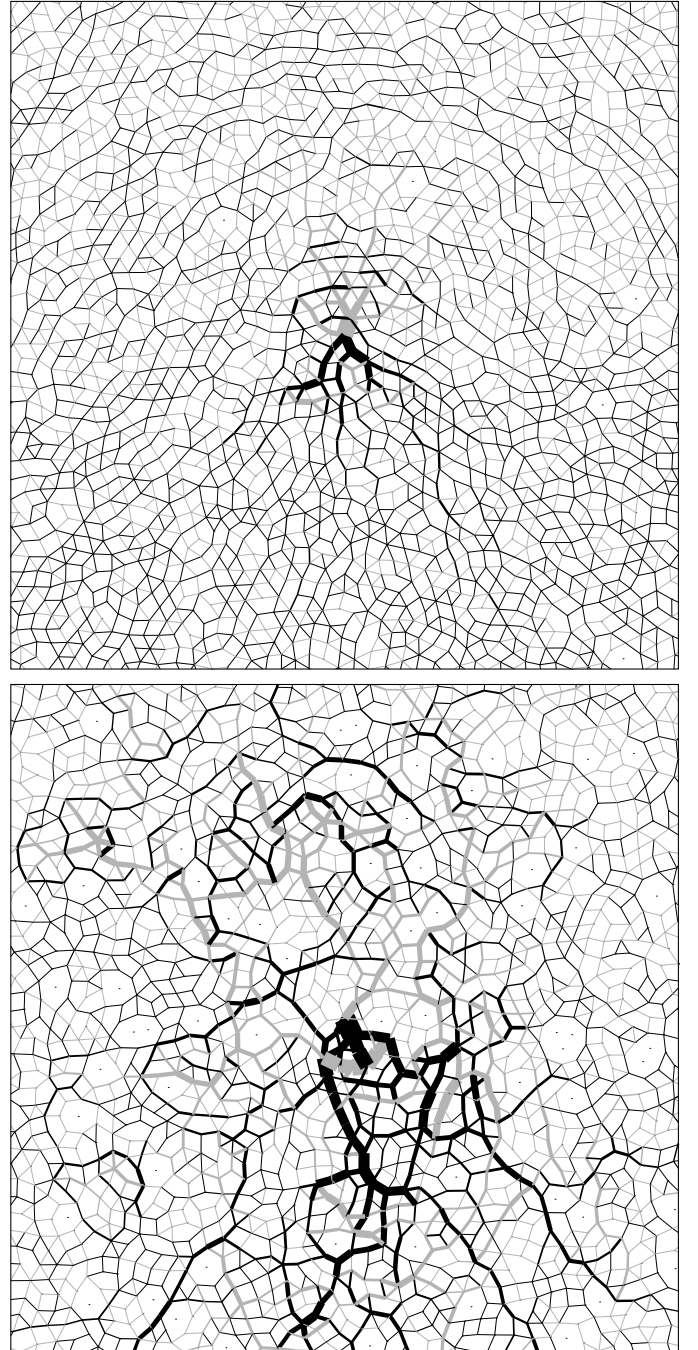


Figure 2: Centred zoom picture of the response to an in-plane external force on a single particle. The upper image is at $p = 10^{-3}$, the lower at $p = 10^{-6}$. Black (grey) lines indicate an increase (decrease) in contact force, the thickness corresponds to the magnitude of the change.

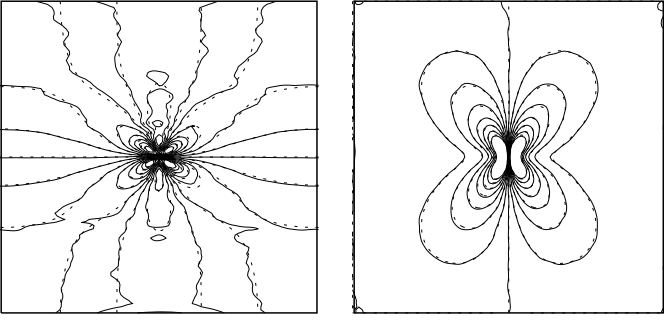


Figure 3: Comparing the average stress response of a few hundred granular packings ($p = 10^{-3}$, dotted) to analytic elasticity calculations ($\nu = 0.62$, solid). The left panel displays contours for σ_{xx} , the right for σ_{xy} .

2.3 Results for point loading

For a unit force in the negative y -direction on a particle in the middle of the packing, the response networks are shown in Figure 2, for a high pressure and a low pressure packing. Comparison of these graphs shows that the fluctuations become larger when the pressure decreases, and spread out over longer distances. We will analyse this in more detail elsewhere and focus here on the average elastic properties.

From the response networks we calculate the changes in the stress tensor, coarse grained over a length scale of the order of one grain diameter (Goldhirsch and Goldenberg 2002). Averaged over many realisations, this stress response field compares very well to analytic calculations of two-dimensional elasticity. In Figure 3 this is shown for packings that were generated at $p = 10^{-3}$. In such a comparison, the poisson ratio ν used in the analytic calculations is the only adjustable parameter. We find that it can be extracted very precisely from these fits. It changes with pressure, approaching unity as $p \rightarrow 0$, which is the maximum value in 2D elasticity. For lower pressures the fluctuations are larger (see also Figure 2) so the obtained ν is slightly less accurate, but the average still fits elastic behaviour quite well.

From a comparison of the displacement field of the granular packing and the analytic calculation we can find the Young modulus Y of the packing in a similar way. A suitable quantity for doing this is the average y -displacement of all particles in a narrow strip at a certain height y

$$V(y) = \frac{1}{N_{\text{strip}}} \sum_{\{i|y_i \approx y\}} u_y^{(i)}, \quad (2)$$

the analytic counterpart of which looks like

$$V(y) = \frac{1}{L_x} \int_0^{L_x} u_y(x, y) dx, \quad (3)$$

where the origin is in the lower left corner, $u_y(x, y)$ is the vertical component of the displacement field, and

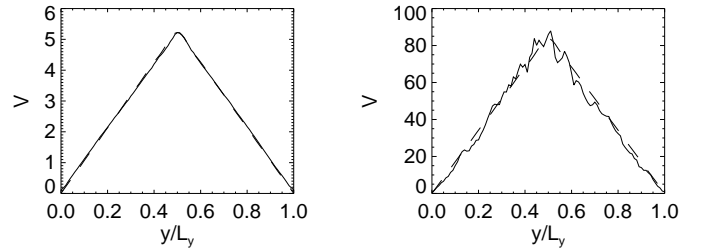


Figure 4: The average vertical displacements of all particles around height y (solid) and the analytic fit (dashed), for $p = 10^{-3}$ (left) and $p = 10^{-6}$ (right).

L_x is the width of the system, The Young modulus appears in this function as an overall multiplicative constant, which makes it easy to fit, as is shown in Figure 4.

Y and ν determine the properties of an isotropic linear elastic medium. The bulk modulus K and shear modulus μ can be calculated from these using the following relations:

$$K = \frac{Y}{2(1-\nu)}, \quad (4)$$

$$\mu = \frac{Y}{2(1+\nu)}. \quad (5)$$

All these quantities are plotted in Figure 5, as a function of pressure. They are found to scale as power laws of p as $p \rightarrow 0$:

$$Y \propto p^{0.60 \pm 0.01}, \quad (6)$$

$$(1-\nu) \propto p^{0.23 \pm 0.03}, \quad (7)$$

$$K \propto p^{0.38 \pm 0.02}, \quad (8)$$

$$\mu \propto p^{0.63 \pm 0.01}. \quad (9)$$

The exponents for K and μ match those found by different methods by O'Hern *et al.* for various (2D, 3D, mono- and bidisperse) systems, including Hertzian spheres (O'Hern *et al.* 2003).

3 OUTLOOK

Averages over many packings have provided us with information about the global elastic properties of granular packings. However, single experiments show fluctuations around this average. The size and range of these fluctuations depend on pressure, and presumably the range of these fluctuations is one of the important length scales which is found to diverge upon approaching point J. We have various indications that to identify this scale, it is useful to consider a global shear as external force, instead of the localized point force. This is illustrated by Figure 6. The higher pressure picture (on the left) is more homogeneous, and

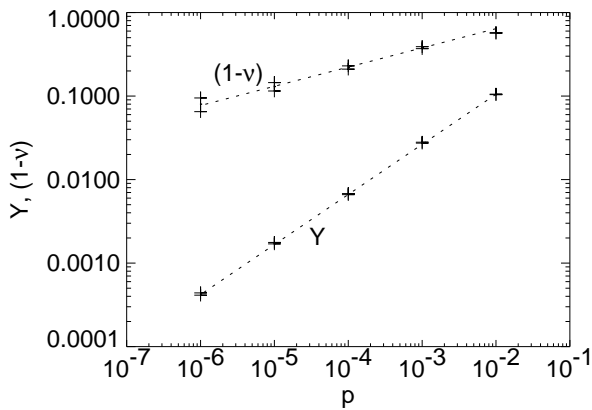
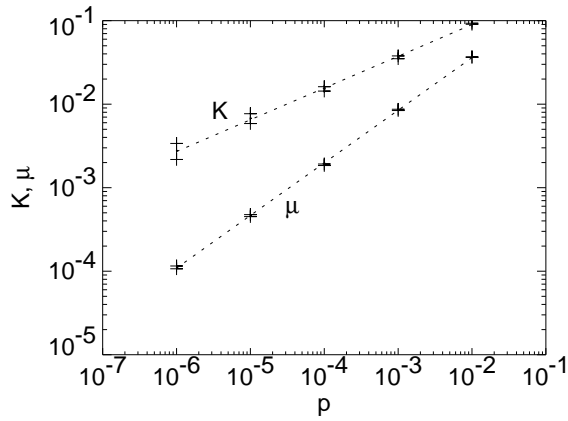


Figure 5: Scaling of the elastic moduli with pressure. The dotted lines are Equations 6 to 9. Bulk, shear and Young moduli are all expressed in units of the Young modulus of the constituent particles.

looks like an elastic response already on the scale of a few grains. In the low pressure case, one would have to average over a larger scale to see a smooth response.

4 CONCLUSIONS

The linear response calculations provide us with a displacement field and changes in contact forces for given external loads on granular packings. We have shown that these can be used to extract the global elastic moduli for these systems. The elastic moduli vanish with pressure as power laws, with exponents that are consistent with earlier work. We will show elsewhere that many of these features are also observed for frictional particles, and that also the vibrational spectra change strongly as a function of pressure.

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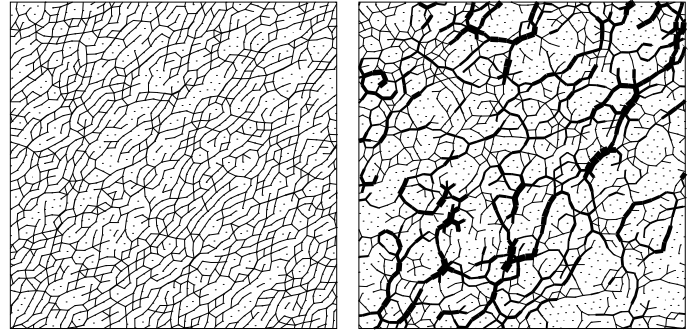


Figure 6: Zoom picture of the response to a global shear, with only the compressed contacts drawn. The left image is at $p = 10^{-3}$, the right at $p = 10^{-6}$.

REFERENCES

- Goldhirsch, I. and C. Goldenberg (2002). On the microscopic foundations of elasticity. *Eur. Phys. J. E* **9**, 245–251.
- Leonforte, F., A. Tanguy, J.P. Wittmer, and J.-L. Barrat (2004). Continuum limit of amorphous elastic bodies II: Linear response to a point source force. *Phys. Rev. B* **70**, 014203.
- Liu, A.J. and S.R. Nagel (1998). Nonlinear dynamics: Jamming is not just cool anymore. *Nature* **396**, 21.
- O’Hern, C.S., L.E. Silbert, A.J. Liu, and S.R. Nagel (2003). Jamming at zero temperature and zero applied stress: The epitome of disorder. *Phys. Rev. E* **68**, 011306.
- Somfai, E., J.-N. Roux, J.H. Snoeijer, M. van Hecke, and W. van Saarloos (2004). Wave propagation in confined granular systems. [cond-mat/0408128](https://arxiv.org/abs/cond-mat/0408128).