Particle Physics of the Early Universe

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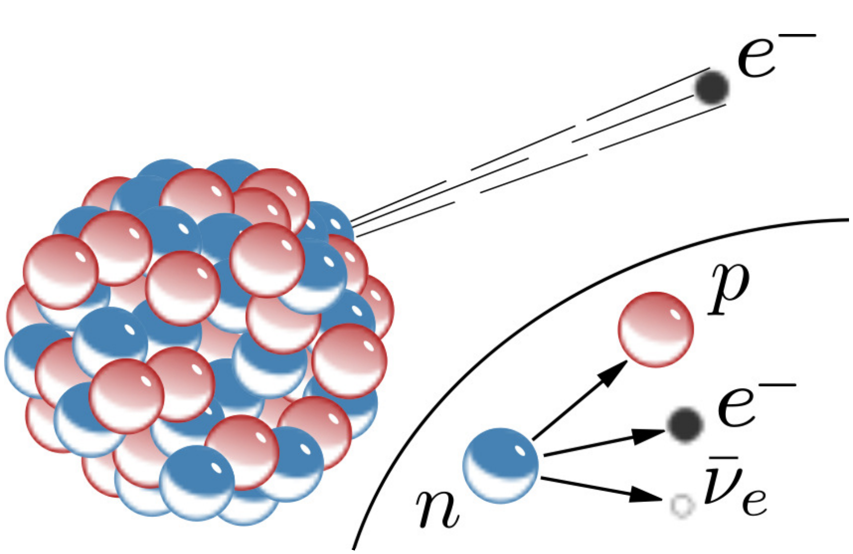
Spring semester 2015

Fermi theory of β-decay [1]



- Neutron decay $n \to p + e^- + \bar{\nu}_e$

- Two papers by E. Fermi:

  An attempt of a theory of beta radiation. 1. (In German) Z.Phys. 88 (1934) 161-177 DOI: 10.1007/BF01351864

  Trends to a Theory of beta Radiation. (In Italian) Nuovo Cim. 11 (1934) 1-19 DOI: 10.1007/BF02959820

Continuum spectrum of electrons (1927)

Prediction of neutrino (1930, 1934)

Fermi theory (1934)

Universality of Fermi interactions (1949)

- Fermi 4-fermion theory:

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{\sqrt{2}}[\bar{p}(x)\gamma_\mu n(x)][\bar{e}(x)\gamma^\mu \nu(x)] \qquad (1)$$

[0]History of β-decay (see **[hep-ph/0001283]**, Sec. 1,1); Cheng & Li, Chap. 11, Sec. 11.1)

Neutrino-electron scattering

- Fermi Lagrangian includes leptonic and hardronic terms:

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{\sqrt{2}}\left[J^\dagger_{\text{lepton}}(x) + J^\dagger_{\text{hadron}}(x)\right] \cdot \left[J_{\text{lepton}}(x) + J_{\text{hadron}}(x)\right] \quad (2)$$

where current J^μ has leptonic ($e^\pm$, $\mu^\pm$, ν, $\bar{\nu}$) and hadronic (p, n, π) parts

- Fermi theory predicts **lepton-only** weak interactions, such as $e + \nu_e \to e + \nu_e$ scattering

$$\mathcal{L}_{\nu e} = \frac{4G_F}{\sqrt{2}}(\bar{e}\gamma_\lambda\nu_e)(\bar{\nu}_e\gamma^\lambda e) \quad (3)$$

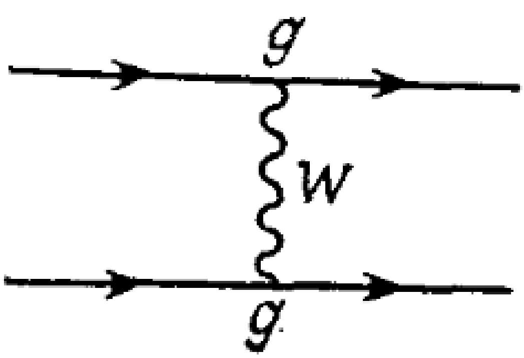
- Matrix element for $e + \nu_e \to e + \nu_e$ scattering

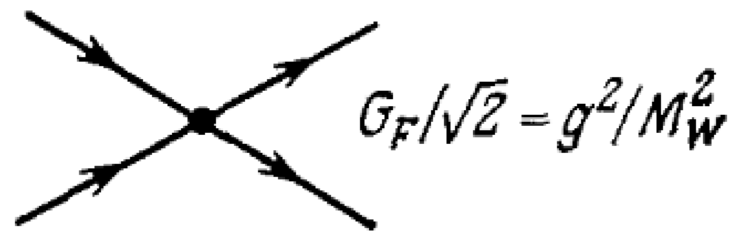
$$\sum_{\text{spins}} |\mathcal{M}|^2 \propto \boxed{G_F^2 E^4_{c.m.}} \quad (4)$$

Massive intermediate particle

Unitarity means that for any process the matrix element should be **bounded from above**: $|\mathcal{M}| \leq \text{const}$

- The matrix element for $e + \nu_e \to e + \nu_e$ scattering grows with energy, $|\mathcal{M}|^2 \propto G_F^2 E_{c.m.}^4$.
- Therefore, the Fermi theory would predict meaningless answers for scattering at energies $E_{c.m.} \gtrsim \sqrt{G_F} \approx 300\,\mathrm{GeV}$
- Promote point-like 4-fermion Fermi interaction to interaction, mediated by a **new massive particle** :





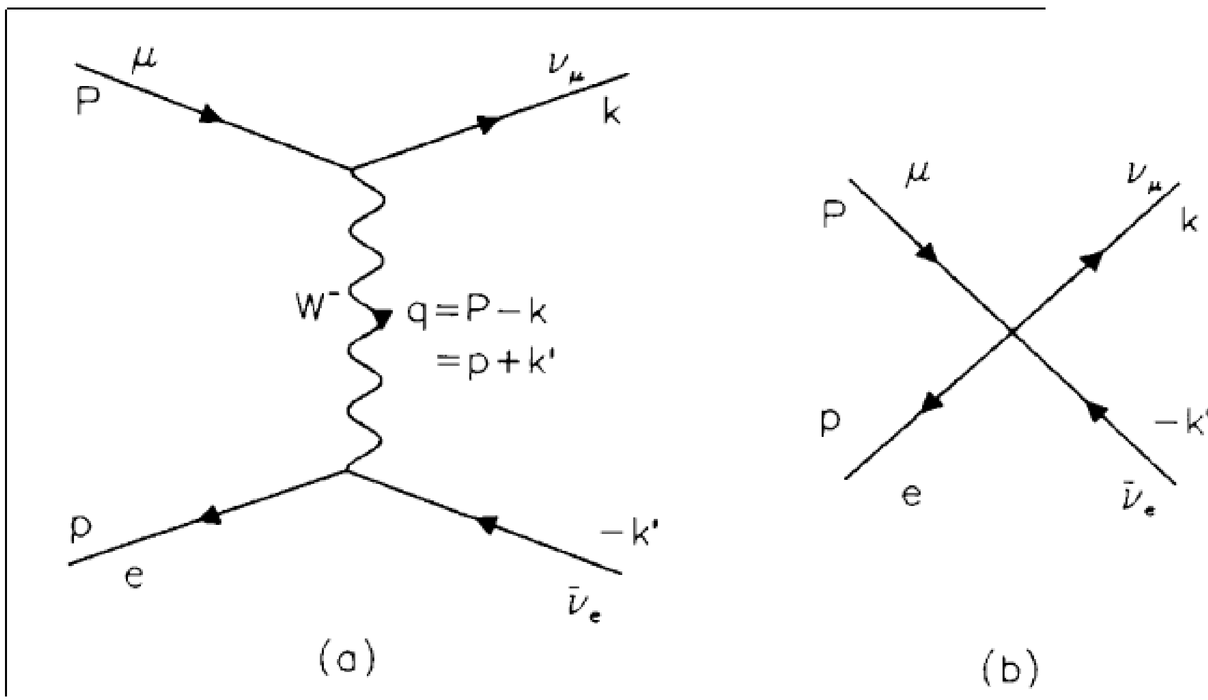
- W – massive particles with M_W

- For $\sqrt{s} \ll M_W$ – looks like a point, 4-fermion interaction
- For $\sqrt{s} \gg M_W$ – behaves as s^{-1}
- What is the spin of this particle? It should couple to **current** so it is a vector field:

$$\mathcal{L}_{\text{int}} = \frac{g}{2\sqrt{2}}\left(J_\mu^+ W_\mu^- + J_\mu^- W_\mu^+\right) \quad (5)$$

g – new coupling constant responsible for weak interactions

- The currents $J_\mu^\pm$ made of electron and ν_e (or muon and ν_μ) carries charge ± 1.



- **Free** massive vector boson obeys Proka equation:

$$\partial_\nu(\partial^\mu W^\nu - \partial^\nu W^\mu) = M_W^2 W^\mu \tag{6}$$

- Taking ∂_μ derivative of (4) $\Rightarrow M_W^2 \partial_\mu W^\mu = 0$
- Eq. (6) can be rewritten as a set of Klein-Gordon equations

$$\Box\, W^\mu = M_W^2 W^\mu \tag{7}$$

- Three independent plane wave solution (because now $W^0 \neq 0$):

$$W_\mu = \epsilon_\mu^{(i)} e^{ix\cdot k} \quad \text{where} \quad \begin{cases} k^\mu \cdot \epsilon_\mu^{(i)} = 0, & i = 1,2,3 \\ k^\mu \cdot k_\mu = M_W^2 \end{cases}$$

$\epsilon_\mu^{(i)}$ are 3 vectors of polarizations

- Consider W boson at rest

- The three polarization vectors are just three unit vectors along the axes x, y, and z

- Now, make boost along z axis. W-bosons 4-momentum $k^\mu = (E, 0, 0, k_z)$

- Three polarization vectors are

$$\begin{cases} \epsilon_\mu^{(1)}(k) = (0, 1, 0, 0) \\ \epsilon_\mu^{(2)}(k) = (0, 0, 1, 0) \\ \epsilon_\mu^{(3)}(k) = \frac{1}{M_W}(k_z, 0, 0, E) \end{cases}$$

- Consider W boson with energy $E \gg M_W$, then $k^\mu = (E, 0, 0, k_z) \approx E(1, 0, 0, 1)$ when $E \approx k_z \gg M_W$

Theory of massive vector boson

- The third polarization vector: $\epsilon_\mu^{(3)} = \frac{E}{M_W}(k_z, 0, 0, E) \approx \frac{E}{M_W}(1, 0, 0, 1)$

$$\epsilon_L \approx \frac{k^\mu}{M_W}$$

- There is a subtle difference here! k^μ is **time-like vector** and ϵ_L is a **space-like vector**. In the relativistic limits they both approach light-cone, but from two different sides

- Longitudinally polarized W_μ in the limit $E \gg M_W$ looks like a derivative of a scalar function $W_L^\mu \approx \frac{1}{M_W}\partial^\mu \phi$

- Interaction with currents:

$$g J_\mu^- W_\mu^+ \xrightarrow{\text{longitudinal}} g \frac{\partial_\mu \phi}{M_W} J_\mu^- = \frac{g}{M_W} \phi(\partial^\mu J_\mu^-)$$

- ...this looks like a new **dimensionful** coupling constant

Massive vector field and Stückelberg field

- Introduce new scalar field, θ. It interacts with a gauge field W, satisfying Maxwell's equation:

$$\partial_\nu(\partial^\mu W^\nu - \partial^\nu W^\mu) = 0 \tag{8}$$

- Under gauge transformation $W_\mu \to W_\mu + \partial_\mu\lambda$ the field θ shifts as $\theta \to \theta - \lambda$

- Equation of motion for θ ($D_\mu\theta = \partial_\mu\theta + W_\mu$):

$$\partial_\mu(D^\mu\theta) = \partial_\mu(\partial^\mu\theta + W^\mu) = 0 \tag{9}$$

- The full equation for W becomes:

$$\partial_\nu(\partial^\mu W^\nu - \partial^\nu W^\mu) = M_W^2 D^\mu\theta \tag{10}$$

- Gauge condition $\theta = 0$ reduces these two equations for W and θ to the old Proka equation

- **Show** that if $\epsilon_\mu^{(i)}$ are 3 polarization vectors than $\sum_{i=1}^3 \epsilon_\mu^{(i)}\epsilon_\nu^{(i)} = \left(-\eta_{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}\right)$
- Define a propagator of massive Klein-Gordon equation with additional condition $\partial_\mu W^\mu = 0$:

$$\langle W_\mu(x)W_\nu(x')\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-x')} \frac{1}{p^2 - M_W^2} \sum_{\text{polarizations}} \epsilon_\mu^i(p)\epsilon_\nu^i(p)$$

$$= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \frac{\eta_{\mu\nu} - \frac{p_\mu p_\nu}{M_W^2}}{p^2 - M_W^2} \tag{11}$$

- Try to put $M_W \to 0$. Will you recover photon-like propagator? **No!** Trouble with the term in the numerator

- The relevant part of the interaction Lagrangian is

$$\Delta\mathcal{L} = gW_\mu^+ \frac{1}{\sqrt{2}} \bar{\nu}_e \gamma^\mu P_L e + gW_\mu^- \frac{1}{\sqrt{2}} \bar{e} \gamma^\mu P_L \nu_e \tag{12}$$

$P_L = \frac{1}{2}(1-\gamma_5)$ – projector of the 4-component spinor on the left chirality.

- The relevant matrix element is given by

$$i\mathcal{M} = \left(\frac{ig}{\sqrt{2}}\right)^2 [\bar{u}(k_1)\gamma^\mu P_L u(p_2)]\,(-i)\left(\frac{g_{\mu\nu} - \frac{r_\mu r_\nu}{M_W^2}}{r^2 - M_W^2}\right)[\bar{u}(k_2)\gamma^\nu P_L u(p_1)] \tag{13}$$

- After average over polarization of the colliding electron and summation over polarizations of other particles we obtain:

$$|\bar{\mathcal{M}}|^2 = \frac{g^4 s^2}{2(r^2 - M_W^2)^2}, \quad s = (p_1 + p_2)^2 \tag{14}$$

- At low energies, when $s, r \ll M_W^2$, but nevertheless $s, r \gg m_e^2$, we get the result of the Fermi theory $|\bar{\mathcal{M}}|^2 = 16 G_F^2 s^2$, provided that we identify

$$G_F = \frac{\sqrt{2} g^2}{8 M_W^2} \tag{15}$$

- In the general case

$$|\bar{\mathcal{M}}|^2 = \frac{2g^4}{\left(1 + \cos\theta + \frac{2M_W^2}{s}\right)^2}, \quad r^2 = (p_2 - k_1)^2 = -\frac{s}{2}(1 + \cos\theta) \tag{16}$$

- The unitarity requirement $\mathcal{M}^{(j)} \leq 1$ then leads to

$$s \leq M_W^2 \left[\exp\left(\frac{16\pi}{g^2}\right) - 1\right] \tag{17}$$

For the known value $\alpha_W = g^2/4\pi \approx 1/30$, we get $\sqrt{s} \lesssim 10^{28}\mathrm{GeV}$.

- **Recall** that the Lagrangian of the massive vector field B_μ would have the form (**Proka Lagrangian**):

$$\mathcal{L}_{\text{Proka}} = -\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}m_B^2 B^\mu B_\mu \tag{18}$$

- **However** W-boson is charged! Under the gauge transformation:

$$W_\mu^\pm \to e^{\pm i\alpha} W_\mu^\pm \tag{19}$$

Therefore, we should change $\partial_\mu \to D_\mu$ in Eq. (18), where

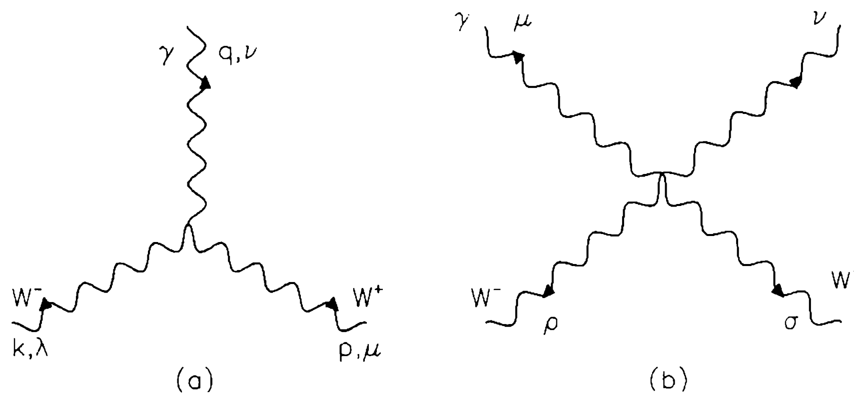
$$D_\mu W_\nu^\pm = (\partial_\mu \pm i\, e\, A_\mu) W_\nu^\pm$$

We follow largely the book by J. Horejsi *"Introduction to Electroweak Unification: Standard Model from Tree Unitarity", Chapters 3 and 4.*

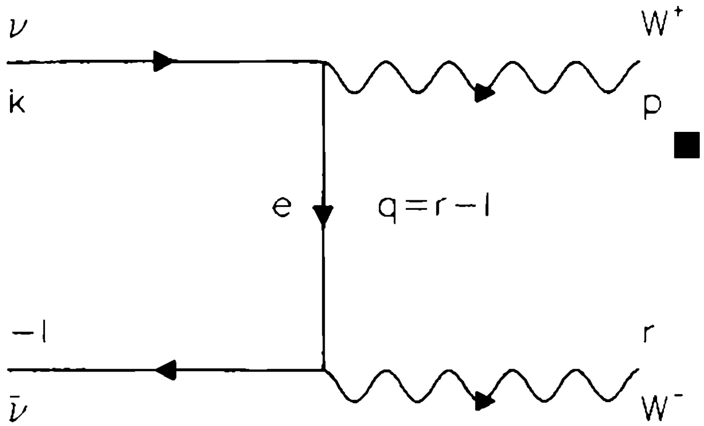
- The kinetic term for W-boson is therefore

$$\mathcal{L}_W = -\frac{1}{2}(D^+_\mu W^-_\nu - D^+_\mu W^-_\nu)(D^-_\mu W^+_\nu - D^-_\mu W^+_\nu) + \frac{M_W^2}{2} W^+_\mu W^-_\mu \quad (20)$$

- **Show** that in addition to (18) Lagrangian (20) also contains terms of the form $W^+W^-\gamma$ **and** $W^+W^-\gamma\gamma$.



- **Write** down explicit form of the $WW\gamma$ and $WW\gamma\gamma$ interactions



- The amplitude of the process:

$$i\mathcal{M} = \bar{v}(p_1)\left(\frac{ig}{\sqrt{2}}\gamma^\mu P_L\right)\frac{i\not{q}}{q^2}\left(\frac{ig}{\sqrt{2}}\gamma^\nu P_L\right)u(p_2)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$$

- Take $s \gg m_e^2$ (so that mass of electron was neglected in the fermion propagator).

$$|\bar{\mathcal{M}}|^2 = \sum_s |\mathcal{M}|^2 = \tag{21}$$

$$= \frac{g^2}{4q^2q^2}\mathrm{Tr}\left[\not{p}_1\,\gamma^\mu\not{q}\,\gamma^\nu\not{p}_2\,\gamma^\rho\not{q}\,\gamma^\lambda\right]\left(-g_{\mu\lambda} + \frac{k_{1\mu}k_{1\lambda}}{M_W^2}\right)\left(-g_{\nu\rho} + \frac{k_{2\nu}k_{2\rho}}{M_W^2}\right)$$

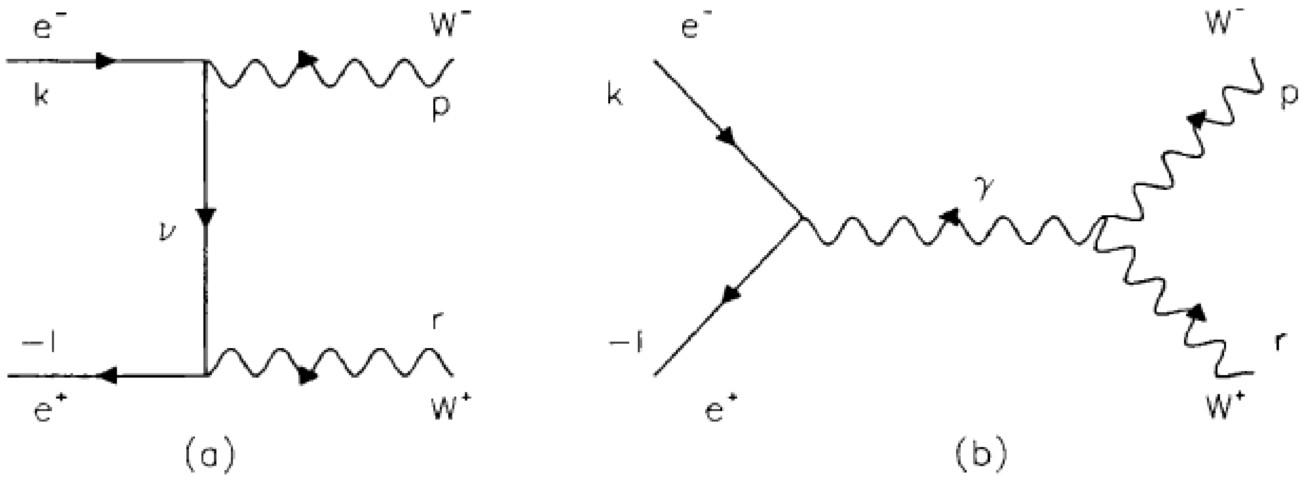
- In the high-energy approximation $s, q^2 \gg M_W$ dominates longitudinal part, coming from $k_{1,2}/M_W$ (i.e. one can neglect $g_{\mu\nu}$ term in the numerator)

$$|\bar{\mathcal{M}}|^2 = \frac{g^2 s^2}{M_W^4} \tag{22}$$

grows as a fourth power of energy.

- Therefore, the considered process **still violates unitarity.**
- Similarly the process $e^+ e^- \to W^+ W^-$ violates unitarity

Can this be amended?



- We saw that as W-bosons are charged, there is $WW\gamma$ interaction term. Therefore the real process $ee \to WW$ is described by the sum of two diagrams

- Can these two diagrams give rise to the unitary behavior of the resulting cross-section?

- **No!** (left diagram is parity violating, right diagram is parity conserved!)

New particle is needed

- What could restore the unitarity of $e^+e^- \to W^+W^-$ **and** $\bar{\nu}\nu \to WW$? $\Rightarrow$**new particle**

- New vector boson that couples to electrons **and to neutrinos** in the **parity violating** way and that also couples to W^+W^-.

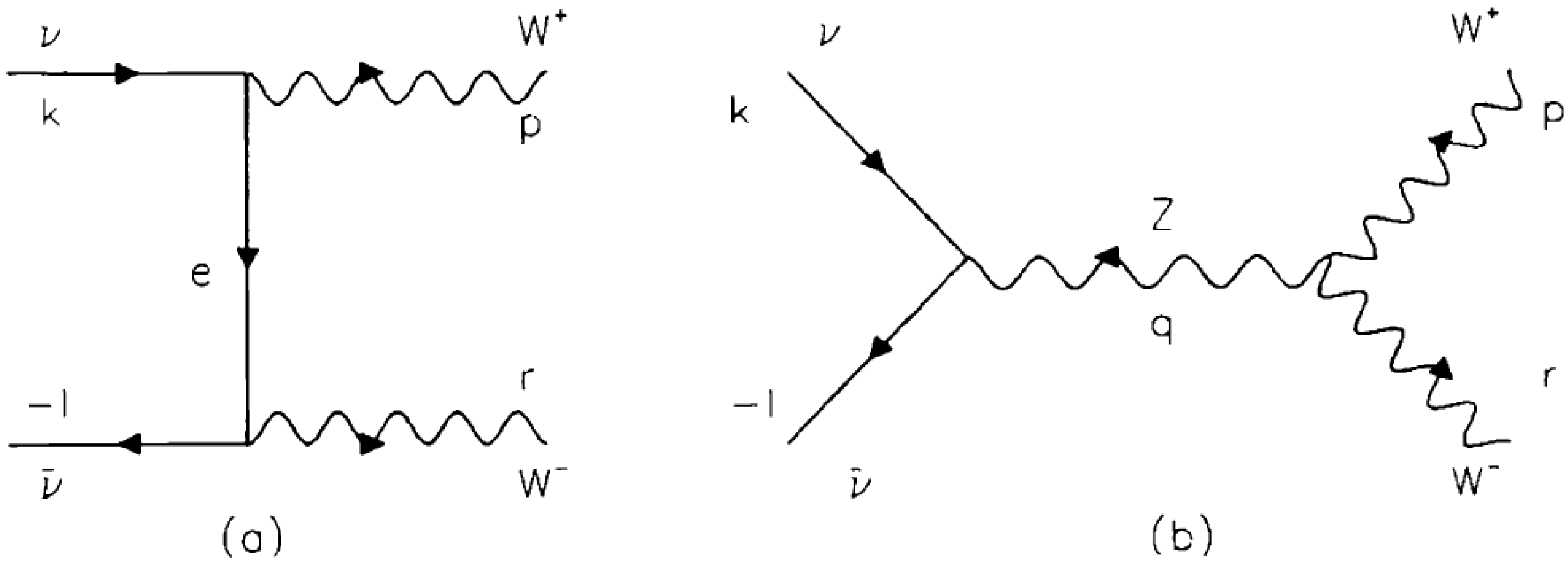
- New boson (Z**-bosons**) interacts with ν:

$$\mathcal{L}_{\bar{\nu}\nu Z} = \frac{1}{2} g_{\bar{\nu}\nu Z} \bar{\nu}\gamma^\mu(1-\gamma_5)\nu Z_\mu \qquad (23)$$

- New boson also interacts with W^+W^- and the vertex WWZ is similar to the vertex $WW\gamma$

- As result there are two processes contributing to $\bar{\nu}\nu \to W^+W^-$ scattering

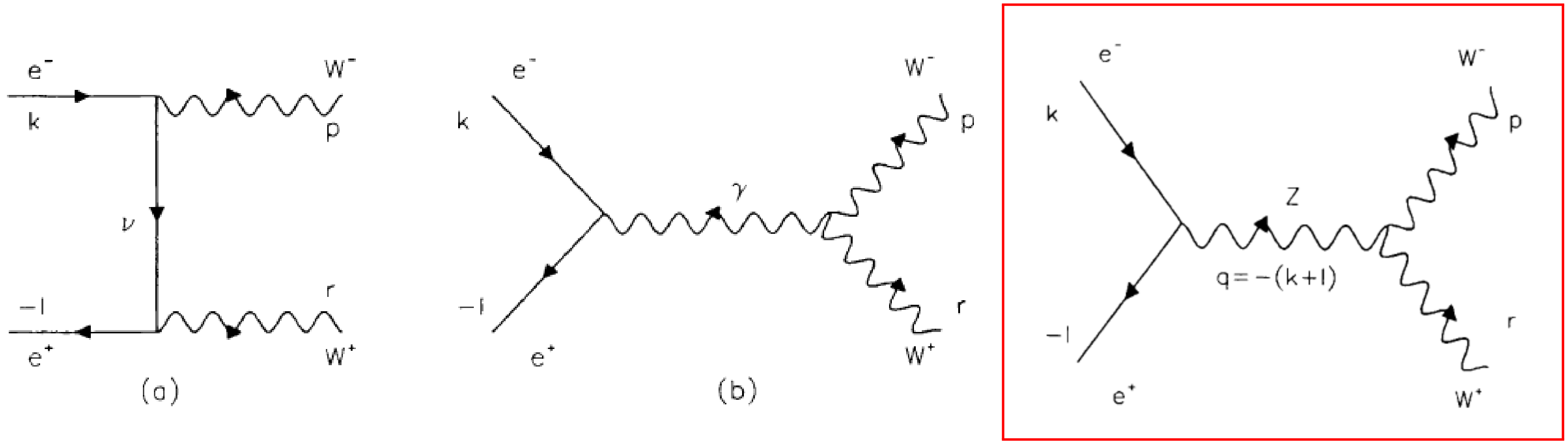
New particle is needed



- As discussed before $\mathcal{M}_a \propto g^2 \left(\frac{E}{M_W}\right)^2$ ← from longitudinally polarized final W states

- $\mathcal{M}_b \propto (g_{\bar{\nu}\nu Z}\ g_{ZWW}) \left(\frac{E}{M_W}\right)^2$ Can cancel contribution of $\mathcal{M}_a$ if $g_{\bar{\nu}\nu Z} g_{ZWW} = \frac{1}{2} g^2$

- Similarly, for $e^+e^- \to W^+W^-$ we would have **three** contributions to the matrix element: $|\mathcal{M}|^2 = \left|\mathcal{M}_a + \mathcal{M}_b + \boldsymbol{\mathcal{M}_c}\right|^2$



- Interaction with electrons:

$$V_{\text{int}} = g_L \bar{e}_L \gamma^\mu e_L Z_\mu + g_R \bar{e}_R \gamma^\mu e_R Z_\mu \tag{24}$$

New symmetry?

- Neutrino and electron – different charge. Different mass.
- . . . **but!** W-boson converts $e \to \nu_e$ or vice versa
- Neutron and proton – different charge. Different mass.
- . . . **but!** W-boson converts $e \to \nu_e$ or vice versa
- **Wild guess:**

> Is there a new symmetry, that “rotates” e into ν_e
> (also μ into ν_μ, p into n, etc.)

- Let we now have 2 different fermion fields $\psi^{(1)}$ and $\psi^{(2)}$ which are physically equivalent for some interaction (good historical example is n and p for strong interactions).

- The Dirac equations are

$$(i\gamma^\mu\partial_\mu - m)\,\psi^{(1)} + V_{int}(\psi^{(1)}) = 0 \tag{25}$$

$$(i\gamma^\mu\partial_\mu - m)\,\psi^{(2)} + V_{int}(\psi^{(2)}) = 0 \tag{26}$$

- We can compose two-component field $\vec{\Psi} = \begin{pmatrix}\psi^{(1)} \\ \psi^{(2)}\end{pmatrix}$ and rewrite the Dirac equations using $\vec{\Psi}$ as

$$(i\gamma^\mu\partial_\mu - m)\,\vec{\Psi} + V_{int}(\vec{\Psi}) = 0 \tag{27}$$

- Probability

$$P = \int d^3x \left[\bar{\psi}^{(1)} \gamma^0 \psi^{(1)} + \bar{\psi}^{(2)} \gamma^0 \psi^{(2)} \right] = \int d^3x \, \vec{\Psi}^+ \vec{\Psi}$$

and Dirac equation (27) is invariant under global transformations:

$$\vec{\Psi}(x) \to \vec{\Psi}'(x) = U \vec{\Psi}(x) \tag{28}$$

that leaves the "length" of the two-dimensional **isovector** $\vec{\Psi}$ invariant.

- Such transformation is called **unitary transformation** and the matrix U in Eq. is 2×2 complex matrix which obeys conditions $U^+ U = \mathbb{1}$ and $\det(U) = 1$.

PHYSICAL REVIEW VOLUME 96, NUMBER 1 OCTOBER

Conservation of Isotopic Spin and Isotopic Gauge Invariance*

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(Received June 28, 1954)

It is pointed out that the usual principle of invariance under isotopic spin rotation is not consistant with the concept of localized fields. The possibility is explored of having invariance under local isotopic spin rotations. This leads to formulating a principle of isotopic gauge invariance and the existence of a **b** field which has the same relation to the isotopic spin that the electromagnetic field has to the electric charge. The **b** field satisfies nonlinear differential equations. The quanta of the **b** field are particles with spin unity, isotopic spin unity, and electric charge $\pm e$ or zero.

INTRODUCTION

THE conservation of isotopic spin is a much discussed concept in recent years. Historically an isotopic spin parameter was first introduced by Heisenberg[1] in 1932 to describe the two charge states (namely neutron and proton) of a nucleon. The idea that the stable even nuclei contain equal numbers of tl in 1937 Breit, Condon, and Present pointe approximate equality of $p-p$ and $n-p$ inte the ^1S state.[2] It seemed natural to assume equality holds also in the other states availat the $n-p$ and $p-p$ systems. Under such an a

Predictions:

- Iso-spin symmetry $\Rightarrow$ new vector particles
- Charged ($W^{\pm}$) and neutral (Z^0)
- New types of interactions between $W^{\pm}$, Z (and photon)

- Make a local gauge transformation in Eq. (56):

$$\left(i\gamma^\mu\partial_\mu + g\gamma^\mu B'_\mu - m\right)U(x)\vec{\Psi} =$$
$$= U(x)\left(i\gamma^\mu\partial_\mu + \gamma^\mu\left(gU^+B'_\mu U + iU^+\partial_\mu(U)\right) - m\right)\vec{\Psi} \quad (29)$$

- To obtain the initial expression with B_μ:

$$B_\mu = U^+B'_\mu U + \frac{i}{g}U^+\partial_\mu(U) \quad (30)$$

$$B'_\mu = UB_\mu U^+ - \frac{i}{g}\partial_\mu(U)U^+ \quad (31)$$

- Law of gauge transformation of long derivative $D_\mu = \partial_\mu - igB_\mu$:

$$D_\mu \to D'_\mu = U(x)D_\mu U^+(x) \quad (32)$$

$F_{\mu\nu}$ for B_μ field

- For electrodynamics we had:

$$F_{\mu\nu} \sim [D_\mu, D_\nu] = -i\frac{e}{c}(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{33}$$

- In analogy let's calculate $[D_\mu, D_\nu]$:

$$\begin{aligned}[D_\mu, D_\nu] &= [\partial_\mu - igB_\mu, \partial_\nu - igB_\nu] = \\ &= -ig\left(\partial_\mu B_\nu - \partial_\nu B_\mu - ig[B_\mu, B_\nu]\right)\end{aligned} \tag{34}$$

So we can define

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - ig[B_\mu, B_\nu]$$

- In terms of initial fields $B^{(i)}_\mu$ this is

$$F^{(i)}_{\mu\nu} = \partial_\mu B^{(i)}_\nu - \partial_\nu B^{(i)}_\mu + 2g\varepsilon_{ijk}B^{(j)}_\mu B^{(k)}_\nu, \qquad i,j,k = 1,2,3 \tag{35}$$

- From transformation law of long derivative $D_\mu \to D'_\mu = U(x)D_\mu U^+(x)$, so:

$$F_{\mu\nu} \to F'_{\mu\nu} = U(x)F_{\mu\nu}U^+(x) \tag{36}$$

- Notice the difference with electrodynamics. There $F_{\mu\nu}$ **was** gauge invariant (electric and magnetic fields did not change when A_μ was gauge transformed)

- Let us try to make a gauge-invariant term out of (36).

$$\mathrm{Tr}\Big(U(x)F_{\mu\nu}U^+(x)\,U(x)F^{\mu\nu}U^+(x)\Big) = \mathrm{Tr}\Big(F_{\mu\nu}F^{\mu\nu}\Big) \tag{37}$$

Recall that if you have any 3 matrices X, Y, Z, then $\mathrm{Tr}(X\,Y\,Z) = \mathrm{Tr}(Y\,Z\,X) = \mathrm{Tr}(Z\,X\,Y)$

- Equations of motion for the field B_μ can be constructed as Euler-Lagrange equation for the action, starting from the gauge-invariant Lagrangian

$$\mathcal{L}_B = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}\sum_{i=1}^{3} F^{(i)}_{\mu\nu}F^{(i),\mu\nu} \tag{38}$$

- In terms of initial fields $B^{(i)}_\mu$ we can define:

$$F^{(i)}_{\mu\nu} = \partial_\mu B^{(i)}_\nu - \partial_\nu B^{(i)}_\mu + 2g\varepsilon_{ijk}B^{(j)}_\mu B^{(k)}_\nu \tag{39}$$

  Using this definition we can write $F_{\mu\nu} = 2F^{(i)}_{\mu\nu}t_i$ and:

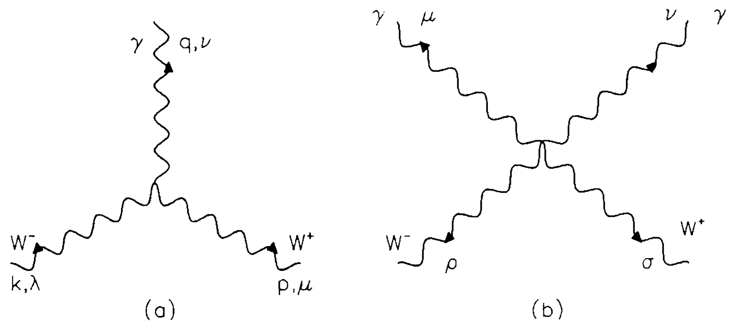
  From full Lagrangian we have:

$$\partial_\mu F^{(i),\mu\nu} = J^{(i),\nu} - 2g\varepsilon_{ijk}B^{(j)}_\mu F^{(k),\mu\nu} \tag{40}$$

  where $J^{(i),\mu} = ig\bar{\Psi}\gamma^\mu t_i\vec{\Psi}$.

- We see, that **even in the absence of matter fields** $B^{(i)}_\mu$ are **self-interacting**.

- Charged nature of the W-bosons leads to two new interaction vertices involving photon



- We deduced some of them from $(D_\mu W_\nu - D_\nu W_\mu)^2$ kinetic term

- Yang-Mills theory predicts another gauge invariant terms, containing **2** W-bosons plus **1** photon or Z boson:

$$\mathcal{L}_{\text{non-minimal}} = e\boldsymbol{\kappa} F^{\mu\nu} W^+_\mu W^-_\nu \tag{41}$$

where $\boldsymbol{\kappa}$ is a dimensionless constant.

notice that the diagram (a) depends both minimal term e that has the structure of $eW\partial WA$ and non-minimal term (41) of the form $e\kappa WW\partial A$

- The interaction vertex $WW\gamma$ has the following form (for the choice of momentum as shown in Fig. (a) above)

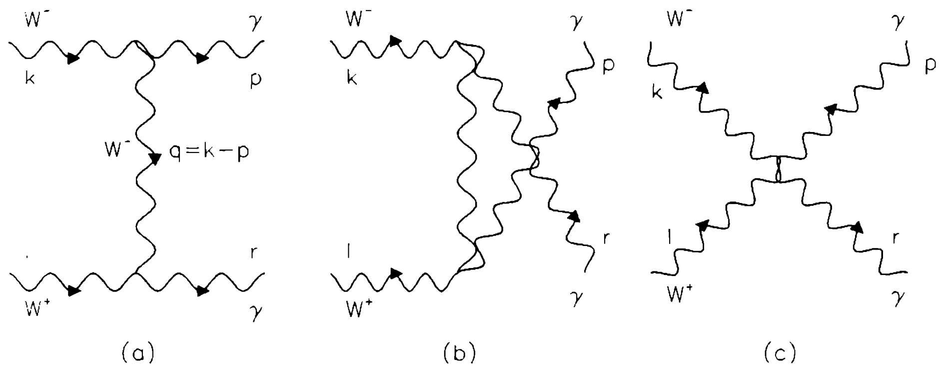
$$V_{\lambda\mu\nu}(k,p,q) = e\underbrace{(k-p)_\nu\eta_{\mu\lambda} + (p-q)_\lambda\eta_{\mu\nu} + (q-k)_\mu\eta_{\nu\lambda}}_{\equiv V^{\text{YM}}_{\lambda\mu\nu}(k,p,q)} + e(1-\kappa)(q_\lambda\eta_{\mu\nu} - q_\mu\eta_{\nu\lambda}) \quad (42)$$

- The interaction vertex $WW\gamma\gamma$ has the structure independent on κ, **but proportional on e^2 rather than e.**

$$V_{\mu\nu\rho\sigma} = -e^2(2\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \quad (43)$$

- **Show** that vertex $V^{\text{YM}}_{\lambda\mu\nu}(k,p,q)$ is symmetric with respect to cyclic change $k \to p \to q$ with the simultaneous $\lambda \to \mu \to \nu$.

- The process $W^+ + W^- \to \gamma + \gamma$ has contribution from both $WW\gamma$ and $WW\gamma\gamma$:



where the first two terms come from the $WW\gamma$ vertex and the last one from $WW\gamma\gamma$.

- ...**keep in mind** that the vertex $WW\gamma$ depends on yet unknown coefficient κ, entering Eqs. (41) and (42) (for example, $\kappa = 0$, i.e. "minimal coupling only" is allowed)

- Let us evaluate the high energy behavior of this process. One can expect that the largest contribution to the diagram comes from the term $\frac{q_\mu q_\nu}{M_W^2}\frac{1}{q^2-M_W^2}$ in the propagator of the virtual W-boson in diagrams (a) and (b).

- Working out the details (see **Horejshi, Eqs. 4.20-4.23** and the text around them) one can see that

> The process $WW \to \gamma\gamma$ violates unitarity at high energies **unless** $\kappa = 1$

- The same is true for the process $\gamma + W \to \gamma + W$

- This special value of $\kappa = 1$ is exactly the one, predicted by the SU(2) gauge invariance!

- For 3 massless fields $B_\mu^{(i)}$ we have 12 components, 3 independent gauge transformations and 3 Gauss laws. So we have 6 degrees of freedom. We will associate those fields with $W^\pm$ and Z bosons.

- But every massless field we have to see at low energies. So our fields have to be massive:

$$\mathcal{L}_B = -\frac{1}{4}\sum_{i=1}^{3} F_{\mu\nu}^{(i)} F^{(i),\mu\nu} - \frac{M_i^2 B_\mu^{(i)} B^{(i),\mu}}{2} \tag{44}$$

- Mass term breaks gauge invariance and gives 3 new longitudinal degrees of freedom.

- In formalism of Stückelberg field we can rewrite longitudinal degrees of freedom at high energies ($E \gg M_i$) using derivatives of scalar field θ:

$$B_\mu^{(i),L} \approx \frac{1}{M_i}\partial_\mu\theta \quad (45)$$

  As we have B^4 term in Lagrangian this means, that at high energies we obtain dimensionful coupling constant and theory become non renormalizable.

- May be, as in electrodynamics, those degrees of freedom can't be excited? No! This field interact non only with conserved current made of fermions (where longitudinal degree of freedom can not be excited, as $J^{(i),\mu}B_\mu^{(i),L} = J^{(i),\mu}\frac{1}{M_i}\partial_\mu\theta = -\partial_\mu J^{(i),\mu}\frac{1}{M_i}\theta = 0$ and $\partial_\mu J^{(i),\mu} = 0$), but also with gauge field contribution to current (self interaction), which is not conserved and, therefore, longitudinal component will be excited! It can not be avoided.

- What to do? How to describe massive vector fields then?

Massive vector bosons and gauge theories

- Electro-magnetic interactions of fermions are ”minimal” interaction i.e. they are required by local (space-time dependent) symmetry – **gauge symmetry**.

- Previously we have see that a self-content theory of weak interactions can by made if these interactions are mediated by 3 massive vector bosons, two charged ($W^{\pm}$ and one neutral (Z).

- Is there a symmetry which requires the existence of $W^{\pm}$ and Z?

- Triplet of intermediate bosons reminds the triplet $\vec{B}_{\mu}$ of Y-M SU(2) fields that we discussed last time.

- However, our fields are massive and, therefore, gauge invariance would be broken. Moreover, the theory of **interacting** massive vector fields is ill-defined, as longitudinal polarisation effectively has dimension-full coupling constant and causes problems.

Spontaneous symmetry breaking

- Let's look at the model of complex scalar field with Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi) \tag{46}$$

where $V(\phi) = m^2 \phi^* \phi + \frac{1}{2}\lambda(\phi^*\phi)^2$ and we denote $|\phi|^2 = \phi^*\phi$.

- This theory is invariant under global U(1) transformation $\boxed{\phi \to \phi' = e^{i\alpha}\phi}$

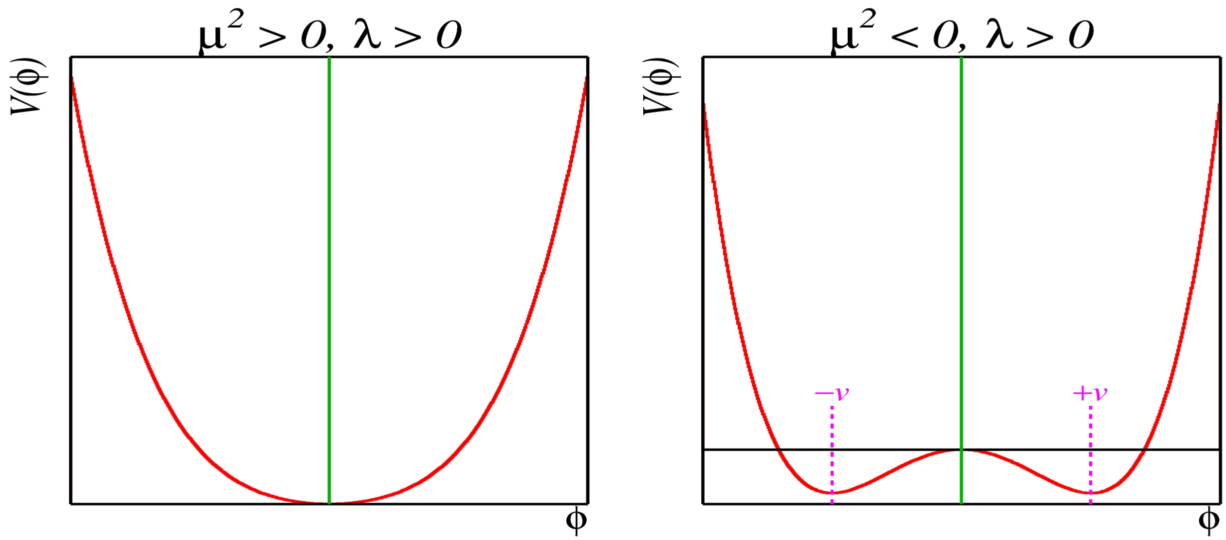
- You can think about complex ϕ as a 2-dimensional vector

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad \text{transforms as} \quad \phi' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$\phi_1 = \mathrm{Re}\,\phi$ and $\phi_2 = \mathrm{Im}\,\phi$

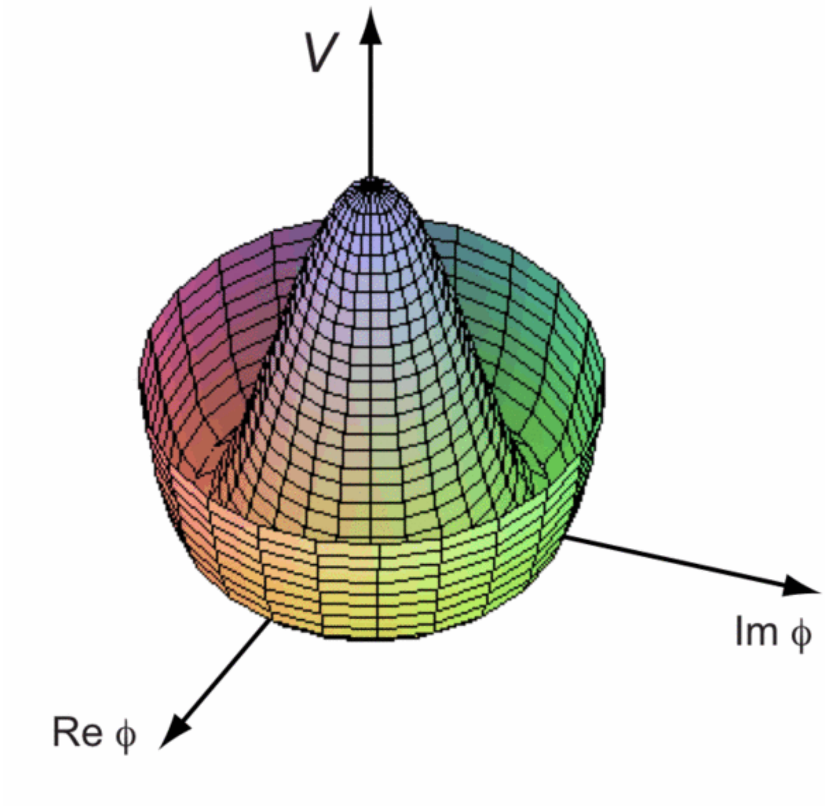
- The **energy density** of this scalar field is

$$E[\phi] = |\dot{\phi}|^2 + |\nabla\phi|^2 + V(\phi) = |\dot{\phi}|^2 + |\nabla\phi|^2 + m^2|\phi|^2 + \frac{1}{2}\lambda|\phi|^4$$



If $m^2 > 0$ this is just a scalar field with the mass m and self-interaction λ

If $m^2 < 0$ the point $\phi = 0$ is the **maximum** of the potential, representing an unstable equilibrium.



There is a true minimum of this potential at $\phi \neq 0$.

$$\frac{\partial V(\phi)}{\partial \phi} = 0 \quad \text{and} \quad \frac{\partial^2 V(\phi)}{\partial \phi^2} > 0 \Longrightarrow$$

$$|\phi|^2_{\text{min}} = -\frac{m^2}{\lambda} > 0$$

Let us also denote the combination $v^2 \equiv -2m^2/\lambda$. This is called **vacuum expectation value** of the field (or **VEV** in short)

If $m^2 < 0$ the potential has **the whole circle of minima** with $|\phi|^2_{\text{min}} = v^2/2$. Any solution of the form $\boxed{\phi = \frac{v}{\sqrt{2}} e^{i\theta}}$ is the minimum .

- Potential is invariant under U(1) transformation $\phi \to \phi' = e^{i\alpha}\phi$
- This transformation rotates one vacuum solution into another

$$\phi = \frac{v}{\sqrt{2}} e^{i\theta} \longrightarrow \phi' = \frac{v}{\sqrt{2}} e^{i(\theta+\alpha)}$$

- Choosing a particular solution (for example, $\phi_{\text{min}} = \frac{v}{\sqrt{2}}$ – real and positive configuration) breaks the symmetry (it is not possible anymore apply $\phi' = e^{i\alpha}\phi$ transformation)
- However, does not change if instead we choose $\phi_{\text{min}} = -\frac{v}{\sqrt{2}}$ or $\phi_{\text{min}} = \frac{v}{\sqrt{2}} e^{i\frac{\pi}{2}}$
- The presence of the spontaneously broken symmetry in the system manifests itself in a special way – as a massless particle.

- Let us compute energy for the configuration $\phi(x) = \frac{v}{\sqrt{2}}e^{i\theta(x)}$. This is not a vacuum, as θ depends on x!

- The potential $V(\phi)$ depends only on $|\phi|$, so $V\left(\frac{v}{\sqrt{2}}e^{i\theta(x)}\right) = -\frac{m^4}{2\lambda}$ — independent on $\theta(x)$. Therefore

$$E\left[\phi(x) = \frac{v}{\sqrt{2}}e^{i\theta(x)}\right] = \frac{v^2}{2}\left(\dot{\theta}^2 + (\nabla\theta)^2\right) - \frac{m^4}{2\lambda}$$

— energy of a **free massless field** with equation of motion $\Box\theta = 0$

- This excitation is called **Goldstone boson**

- The massless field θ describes motion along the circle of minima in the “Mexican hat potential”

- Displace our solution from the minimum in the direction orthogonal to the circle:

$$\phi(x) = \frac{v}{\sqrt{2}}e^{i\theta} + \delta\phi(x) = \frac{1}{\sqrt{2}}\Big(v + \rho(x)\Big) \qquad (47)$$

- The energy now has the form:

$$E[\rho] = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}(\nabla\rho)^2 + \frac{1}{2}\underbrace{v^2\lambda\rho^2}_{\text{mass term for }\rho} + \underbrace{\tfrac{1}{2}v\lambda\rho^3 + \tfrac{\lambda}{8}\rho^4 - \tfrac{\lambda}{8}v^4}_{\text{self-interactions of }\rho}$$

- Oscillations in the direction perpendicular to the circle of vacua are described by the massive **real** scalar field with the mass $m_\rho = v\sqrt{\lambda}$ and equation of motion $(\Box + m_\rho^2)\rho = 0$

- The same result can be seen in a different way.
- Choose one particular minimum, say the one where field value is real and positive ($\phi = v/\sqrt{2}$).
- The symmetry is now **spontaneously broken** – all minima are equivalent and you have chosen a particular one
- expanding about that point:

$$\phi = \frac{1}{\sqrt{2}}(v + \varphi_1 + i\varphi_2) \tag{48}$$

where φ_i are two real scalar fields.

- Then one find:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \varphi_1)^2 + (\partial_\mu \varphi_2)^2\right] - V(\varphi_1, \varphi_2) \tag{49}$$

$$V(\varphi_1, \varphi_2) = -\frac{1}{8}\lambda v^4 + \frac{1}{2}\lambda v^2 \varphi_1^2 + \frac{1}{2}\lambda v \varphi_1(\varphi_1^2 + \varphi_2^2) + \frac{1}{8}\lambda(\varphi_1^2 + \varphi_2^2)^2 \tag{50}$$

- The first term here is merely an unimportant constant.

- The ϕ_1 is the real massive scalar field with the mass $\sqrt{\lambda} v$ (the blue term)

- There is no term quadratic in φ_2! $\Rightarrow \varphi_2$ is a massless field, a **Goldstone boson**.

BROKEN SYMMETRIES AND THE MASSES OF GAUGE BOSONS

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(Received 31 August 1964)

In a recent note[1] it was shown that the Goldstone theorem,[2] that Lorentz-covariant field theories in which spontaneous breakdown of symmetry under an internal Lie group occurs contain zero-mass particles, fails if and only if the conserved currents associated with the internal group are coupled to gauge fields. The purpose of the present note is to report that, as a consequence of this coupling, the spin-one quanta of some of the gauge fields acquire mass; the longitudinal degrees of freedom of these particles (which would be absent if their mass were zero) go over into the Goldstone bosons when the coupling tends to zero. This phenomenon is just the relativistic analog of the plasmon phenomenon to which Anderson[3] has drawn attention: that the scalar zero-mass excitations of a superconducting neutral Fermi gas become longitudinal plasmon modes of finite mass when the gas is charged.

The simplest theory which exhibits this behavior is a gauge-invariant version of a model used by Goldstone[2] himself: Two real[4] scalar fields φ_1, φ_2 and a real vector field A_μ interact through the Lagrangian density

$$L = -\tfrac{1}{2}(\nabla\varphi_1)^2 - \tfrac{1}{2}(\nabla\varphi_2)^2 - V(\varphi_1^{\,2} + \varphi_2^{\,2}) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad (1)$$

where

$$\nabla_\mu\varphi_1 = \partial_\mu\varphi_1 - eA_\mu\varphi_2,$$

about the "vacuum" solution $\varphi_1(x) = 0$, $\varphi_2(x) = \varphi_0$:

$$\partial^\mu\{\partial_\mu(\Delta\varphi_1) - e\varphi_0 A_\mu\} = 0, \qquad (2a)$$

$$\{\partial^2 - 4\varphi_0^{\,2}V''(\varphi_0^{\,2})\}(\Delta\varphi_2) = 0, \qquad (2b)$$

$$\partial_\nu F^{\mu\nu} = e\varphi_0\{\partial^\mu(\Delta\varphi_1) - e\varphi_0 A_\mu\}. \qquad (2c)$$

Equation (2b) describes waves whose quanta have (bare) mass $2\varphi_0\{V''(\varphi_0^{\,2})\}^{1/2}$; Eqs. (2a) and (2c) may be transformed, by the introduction of new variables

$$B_\mu = A_\mu - (e\varphi_0)^{-1}\partial_\mu(\Delta\varphi_1),$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu = F_{\mu\nu}, \qquad (3)$$

into the form

$$\partial_\mu B^\mu = 0, \quad \partial_\nu G^{\mu\nu} + e^2\varphi_0^{\,2}B^\mu = 0. \qquad (4)$$

Equation (4) describes vector waves whose quanta have (bare) mass $e\varphi_0$. In the absence of the gauge field coupling ($e = 0$) the situation is quite different: Equations (2a) and (2c) describe zero-mass scalar and vector bosons, respectively. In passing, we note that the right-hand side of (2c) is just the linear approximation to the conserved current: It is linear in the vector potential, gauge invariance being maintained by the presence of the gradient term.[5]

When one considers theoretical models in which spontaneous breakdown of symmetry under a semisimple group occurs, one encounters a

Higgs vs. Stückelberg models

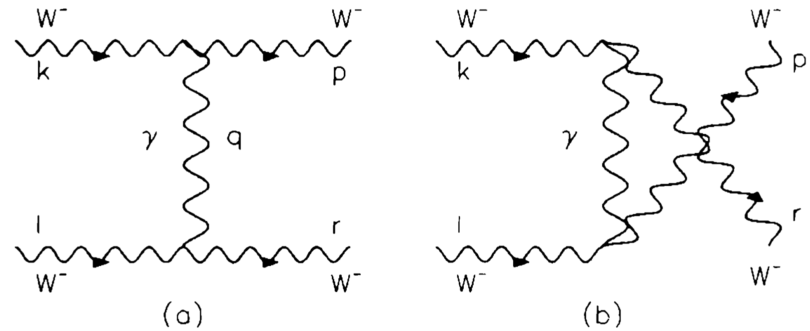
- Models are similar to each other. Both have scalar field that generate longitudinal components for vector fields and give them their masses.

- The main difference is that Higgs model has 2 fields and one of them stays massive. In general, we can write:

$$\phi = \frac{1}{\sqrt{2}}(v + \varphi_1 + i\varphi_2) = \rho e^{i\varphi} \tag{51}$$

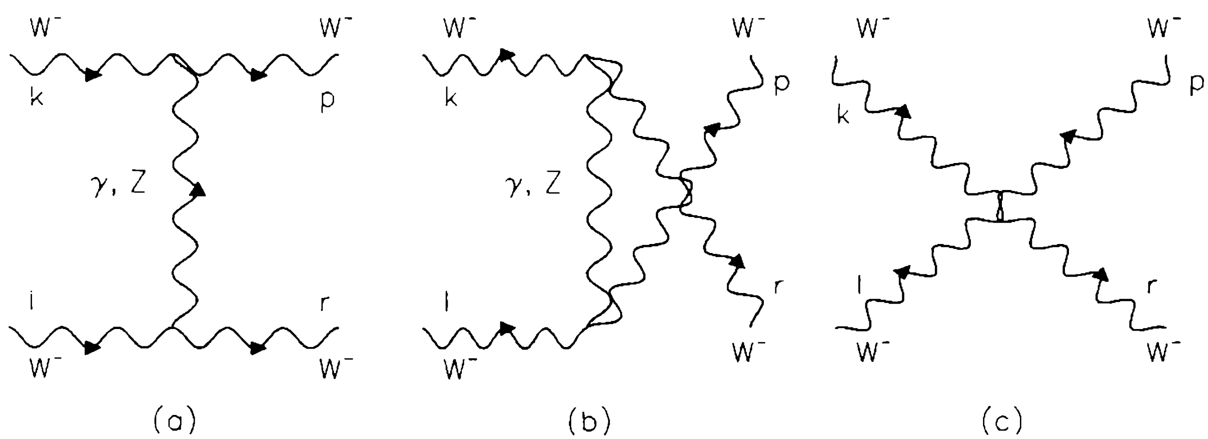
  Here ρ will be a massive field and φ is the full analogue of the Stückelberg field.

- Now let us analyze tree-level $WW \to WW$ scattering, occurring via virtual photon:

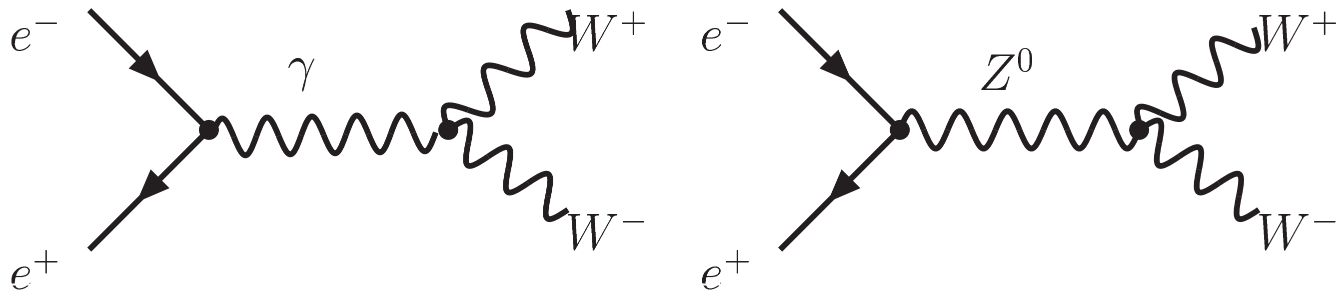


- It can be demonstrated (see Horejshi, Chapter 4.2 (4.25 and till the end of the chapter) that no choice of κ allows to have $|\mathcal{M}| = \mathrm{const}$

- **However** inclusion of Z-boson **and** of additional $W^+W^-W^+W^-$



makes the growth of amplitudes with energy milder



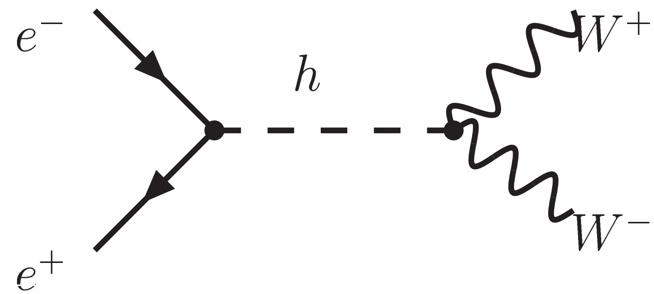
The divergence is in reality not suppressed completely, only in the leading order

$$|\mathcal{M}|^2 \sim g^2 \frac{E^2}{M_W^2} \quad \text{Instead of} \quad |\mathcal{M}|^2 \sim g^2 \frac{E^4}{M_W^4}$$

This happens only for a special choice of Z couplings.

What is the reason? Is there some hidden symmetry behind the choice?

Introduction of a new scalar particle cancels all the residual divergences



This particle is called the **Higgs boson**

Higgs boson is also required to make unitary $WW \to WW$, $ee \to ZZ$ and $WW \to ZZ$

Finally, the theory is **self-consistent**!

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(Received 17 October 1967)

Leptons interact only with photons, and with the intermediate bosons that presumably mediate weak interactions. What could be more natural than to unite[1] these spin-one bosons into a multiplet of gauge fields? Standing in the way of this synthesis are the obvious differences in the masses of the photon and intermediate meson, and in their couplings. We might hope to understand these differences by imagining that the symmetries relating the weak and electromagnetic interactions are exact symmetries of the Lagrangian but are broken by the vacuum. However, this raises the specter of unwanted massless Goldstone bosons.[2] This note will describe a model in which the symmetry between the electromagnetic and weak interactions is spontaneously broken, but in which the Goldstone bosons are avoided by introducing the photon and the intermediate-boson fields as gauge fields.[3] The model may be renormalizable.

- Let us consider for simplicity only the theory containing the e, ν_e (as well γ and intermediate bosons $W^{\pm}, Z$ representing electromagnetic and weak interactions).

- The symmetry that will be ”gauged” (i.e. made space-time dependent) can then be a unitary transformation mixing wave functions of e and ν_e.

- Such a symmetry is $U(2)$ symmetry, contacting 4 generators (3 for the $SU(2)$ part and one common phase for both fermions). This seems to be about right, as we need 4 vector bosons.

- However, e and ν_e have different massed and different electric charges! The same is true for $W^{\pm}$ and Z. Also left and right leptons have different charges under weak interactions

- Therefore, the symmetry should be broken. This would also help to make vector bosons massive.

A MODEL OF LEPTONS*

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Leptons interact only with photons, and with the intermediate bosons that presumably mediate weak interactions. What could be more natural than to unite[1] these spin-one bosons into a multiplet of gauge fields? Standing in the way of this synthesis are the obvious differences in the masses of the photon and intermediate meson, and in their couplings. We might hope to understand these differences by imagining that the symmetries relating the weak and electromagnetic interactions are exact symmetries of the Lagrangian but are broken by the vacuum. However, this raises the specter of unwanted massless Goldstone bosons.[2] This note will describe a model in which the symmetry between the electromagnetic and weak interactions is spontaneously broken, but in which the Goldstone bosons are avoided by introducing the photon and the intermediate-boson fields as gauge fields.[3] The model may be renormalizable.

We will restrict our attention to symmetry groups that connect the observed electron-type leptons only with each other, i.e., not with muon-type leptons or other unobserved leptons or hadrons. The symmetries then act on a left-handed doublet

$$L \equiv [\tfrac{1}{2}(1+\gamma_5)]\binom{\nu_e}{e} \qquad (1)$$

and on a right-handed singlet

$$R \equiv [\tfrac{1}{2}(1-\gamma_5)]e. \qquad (2)$$

The largest group that leaves invariant the kinematic terms $-\bar{L}\gamma^\mu\partial_\mu L - \bar{R}\gamma^\mu\partial_\mu R$ of the Lagrangian consists of the electronic isospin $\vec{T}$ acting on L, plus the numbers N_L, N_R of left- and right-handed electron-type leptons. As far as we know, two of these symmetries are entirely unbroken: the charge $Q = T_3 - N_R - \frac{1}{2}N_L$, and the electron number $N = N_R + N_L$. But the gauge field corresponding to an unbroken symmetry will have zero mass,[4] and there is no massless particle coupled to N,[5] so we must form our gauge group out of the electronic isospin $\vec{T}$ and the electronic hyperchange $Y \equiv N_R + \frac{1}{2}N_L$.

Therefore, we shall construct our Lagrangian out of L and R, plus gauge fields $\vec{A}_\mu$ and B_μ coupled to $\vec{T}$ and Y, plus a spin-zero doublet

$$\varphi = \binom{\varphi^0}{\varphi^-} \qquad (3)$$

whose vacuum expectation value will break $\vec{T}$ and Y and give the electron its mass. The only renormalizable Lagrangian which is invariant under $\vec{T}$ and Y gauge transformations is

- Which symmetries are we going to "gauge"? Let us consider the properties of electric charge in electric + weak interactions.

- Normally, electric charge = number of charged fermions ($N_R + N_L$). But we also have charged vector bosons! How to account for this?

- In weak interactions N_R and N_L does not change, but electric charge of **fermions** can, $e^{\pm} \to W^{\pm} + \nu_e$.

- The number of such events can be counted one of the $su(2)$ generators, T_3. Its eigen values are equal to the difference between numbers of left-handed neutrinos and electrons. Therefore

$$Q = T_3 - N_R - \frac{1}{2} N_L = T_3 - Y \qquad (52)$$

where $Y = N_R - \frac{1}{2} N_L$ is called **hypercharge**, electric charge in the fermionic sector (note $\frac{1}{2}$!)

- Our model must have $U(1) \times SU(2)$ symmetry. Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + g\vec{A}_\mu \times \vec{A}_\nu)^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 +$$
$$+ \bar{R} i\gamma^\mu(\partial_\mu - ig' B_\mu)R + \bar{L} i\gamma^\mu \left(\partial_\mu - ig\vec{t}\vec{A}_\mu - ig'\frac{\mathbf{1}}{\mathbf{2}}B_\mu\right) L -$$
$$- \frac{1}{2}\left|\partial_\mu\phi - ig\vec{t}\vec{A}_\mu + ig'\frac{\mathbf{1}}{\mathbf{2}}B_\mu\right|^2 + \underbrace{m^2\phi^*\phi - \frac{1}{2}\lambda(\phi^*\phi)^2}_{\text{Higgs potential}} \tag{53}$$

B_μ is $U(1)$ field and $\vec{A}_\mu$ is three field transformed by $SU(2)$ group.

- Higgs field is SU(2) **doublet** and **also** has U(1) symmetry (i.e. $\varphi_{1,2}$ are **complex** fields): $\phi = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}$, where $\varphi_{1,2} \in \mathbb{C}$

- Higgs potential has $m^2 < 0$ and therefore there is a **spontaneous symmetry breaking** (as in the previous examples)

- The minimum of its potential is given by the condition:

$$|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 = (\mathrm{Re}\,\phi_1)^2 + (\mathrm{Im}\,\phi_1)^2 + (\mathrm{Re}\,\phi_2)^2 + (\mathrm{Im}\,\phi_2)^2 = -\frac{m^2}{\lambda}$$

- By convention the Higgs field vacuum is chosen in the form

$$\phi = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

where $v^2 = -\frac{2m^2}{\lambda}$ – positive real constant, called **Higgs vev**

- **Only** the combination of $T_3 + Y$ doesn't change Higgs field ($ig\vec{t}\vec{A}_\mu - ig'\frac{1}{2}B_\mu$). **So, there will be 1 massless vector field** out of 4

- In our model physical observable will be combinations of B_μ and $\vec{A}_\mu$ fields:

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\left(A^{(1)}_{\mu} \mp i A^{(2)}_{\mu}\right) \tag{54}$$

$$Z_{\mu} = \frac{1}{\sqrt{g^2 + g'^2}}\left(g A^{(3)}_{\mu} + g' B_{\mu}\right) \tag{55}$$

$$A_{\mu} = \frac{1}{\sqrt{g^2 + g'^2}}\left(-g' A^{(3)}_{\mu} + g B_{\mu}\right) \tag{56}$$

- Masses of these fields are $M_W = \frac{1}{2} g\lambda$, $M_A = 0$, $M_Z = \frac{1}{2}\sqrt{g^2 + g'^2}\lambda$. Also we can find electric charge as $e = gg'/\sqrt{g^2 + g'^2}$. The Fermi constant in this terms is:

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_Z^2} = \frac{1}{2\lambda^2} \tag{57}$$

So we know λ value from G_F. From measuring M_W and e we'll find g and g'. So we can predict M_Z and make an independent experimental cross-check.

- Our model has massless fermions at this model. We can't just write mass terms because of left/right asymmetry (neutrino masses are much smaller than for electrons). Solution is Yukawa mechanism. Let's introduce term:

$$\Delta\mathcal{L} = -G_e\left(\bar{L}\phi R + \bar{R}\phi^+ L\right) \tag{58}$$

Higgs field is two-component: $\phi = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}$. If minimum is in point $\phi_0 = \begin{pmatrix}0 \\ v\end{pmatrix}$, then we obtain massless neutrinos and massive electrons.

- We can measure fermions to Higgs coupling constants G_e in two ways: from measuring fermion masses and from direct experiments with Higgs particle, finding it's decay widths. Even at lower energy Higgs particle give measurable loop corrections. So it is a source of another non-trivial cross-checks of our model.

Additional information

Groups and algebras

- Let matrices U_1, U_2 are such that $U_i^+ U_i = 1$ and $\det(U_i) = 1$. It's easy to see, that matrix $U_1 U_2$ has this properties:

$$(U_1 U_2)^+ U_1 U_2 = U_2^+ U_1^+ U_1 U_2 = U_2^+ \left(U_1^+ U_1\right) U_2 = U_2^+ U_2 = \mathbb{1} \quad (59)$$

$$\det(U_1 U_2) = \det(U_1)\det(U_2) = 1 \quad (60)$$

Besides that, indentity matrix $\mathbb{1}$ and $U_i^{-1} = U_i^+$ also obey this properties.

Notice, that for example, $U_1 + U_2$ is in general not a unitary matrix. Also if U has $\det U = 1$, the matrix $\alpha \times U$ has $\det(\alpha U) = \alpha^n \neq 1$

- This means that set of such matrices is closed under multiplication and inversion operations and has unity element. In math such sets called **groups**, in particular, this group has it's own name — $SU(2)$ group.

- The general element of $SU(2)$ group is parametrized by three **real** numbers and has the form

$$U = \begin{pmatrix} \cos(\theta)e^{i\phi} & -\sin(\theta)e^{-i\beta} \\ \sin(\theta)e^{i\beta} & \cos(\theta)e^{-i\phi} \end{pmatrix} \tag{61}$$

where θ, β, ϕ are three independent angles

- Transformations that are very close to unity (infinitesimal transformations):

$$U = \mathbb{1} + i\delta T,\ \delta \ll 1 \tag{62}$$

where T is some 2×2 matrix

- for example: take $\beta = \phi = 0$ and $\theta \ll 1$ in Eq. (61). Then

$$U \approx \begin{pmatrix} 1 & -\theta \\ \theta & 1 \end{pmatrix} + \mathcal{O}(\theta^2) = 1 + \theta\sigma_x$$

- If $U \in SU(2)$ then:

$$\mathbb{1} = U^+U = \mathbb{1} + i\delta(T - T^+) - \delta^2 T^+ T \tag{63}$$

$$1 = \det(U) = \det(\mathbb{1} + i\delta T) \approx \det(e^{i\delta T}) = e^{i\delta\,\mathrm{Tr}(T)} \tag{64}$$

In linear order by δ, conditions on T matrices are:

$$T = T^+ \quad \text{and} \quad \mathrm{Tr}(T) = 0$$

Set of such matrices is called **Hermitian**

An example of Hermitian matrices are Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$

$$SU(2)$$

- Set of 2×2 Hermitian matrices is called **algebra** of $SU(2)$ group.

- Notice that if T_1 and T_2 are Hermitian matrices, then $T_1 T_2$ is **not** Hermitian in general.

  Indeed, $(T_1 T_2)^+ = T_2^+ T_1^+ = T_2 T_1 \neq T_1 T_2$ (consider σ_x and σ_y as examples!)

  Notice, that sum of Hermitian matrices **is** a Hermitian matrix

- However, if we construct a **commutator**

$$i[T_1, T_2] \equiv i(T_1 T_2 - T_2 T_1) \quad (65)$$

the result of such an operation on two Hermitian matrices is **always** a Hermitian matrix!

$$(i[T_1, T_2])^+ = -i\left((T_1 T_2)^+ - (T_2 T_1)^+\right) = \\ = -i\left(T_2^+ T_1^+ - T_1^+ T_2^+\right) = i[T_1, T_2] \quad (66)$$

$$\mathrm{Tr}(i[T_1, T_2]) = i\,\mathrm{Tr}(T_1 T_2 - T_2 T_1) = i\,\mathrm{Tr}(T_1 T_2) - i\,\mathrm{Tr}(T_2 T_1) = 0 \quad (67)$$

- Algebra of Hermitian matrices is an linear space: if T_1 and T_2 are matrices from algebra, then

$$\alpha T_1 + \beta T_2 \tag{68}$$

is also matrix from algebra for any real α and β.

- Algebra of $SU(2)$ group is 3-dimension linear space, so we can find the basis of this space. Usually this basis is chosen as $t_i = \frac{\sigma_i}{2}$, where σ_i are Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{69}$$

So every element $T = \vec{\alpha} \cdot \vec{t}$. This basis has easy commutation relations:

$$[t_i, t_j] = i\varepsilon_{ijk} t_k \tag{70}$$

- Every element of $SU(2)$ group can be present as infinity number of infinitesimal transformations:

$$U(\alpha_x, \alpha_y, \alpha_z) = \lim_{N\to\infty}\left(\mathbb{1} + \frac{i\vec{\alpha}\vec{t}}{N}\right)^N = e^{i\vec{\alpha}\vec{t}} \qquad (71)$$

So, knowledge of algebra gives all group elements.

- 2×2 unitary matrices is not the only representation for $SU(2)$ group. The same algebra we obtain for 3-dimension rotations group:

$$\tilde{t}_1 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \tilde{t}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \tilde{t}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \qquad (72)$$

For them $[\tilde{t}_i, \tilde{t}_j] = i\varepsilon_{ijk}\tilde{t}_k$ and we can obtain any element of rotation group using exponentiation.

- We saw that probability and other observables do not change if we interchange two fermions $\psi^{(1)}$ and $\psi^{(2)}$.

- Can we make the SU(2) transformation local (i.e. $\vec{\Psi}' \to U(x)\vec{\Psi}$)?

- We can write $U(x) = e^{i(\vec{\alpha}(x)\vec{t})}$ and after local transformation $\vec{\Psi}(x) \to \vec{\Psi}'(x) = U(x)\vec{\Psi}(x)$ we have:

$$U(x)\left(i\gamma^\mu\partial_\mu - \gamma^\mu\partial_\mu(\vec{\alpha}(x))\vec{t} - m\right)\vec{\Psi} = 0 \tag{73}$$

- As we have 3 independent functions $\alpha_i(x)$ we need at least 3 fields to make the Equation (73) invariant.

- In analogy to QED we introduce 3 vector fields, $B_\mu^{(1)}$, $B_\mu^{(2)}$ and $B_\mu^{(3)}$. The main innovation is the way in which these fields interact:

$$V_{int} = g\bar{\Psi}\gamma^\mu B_\mu^{(i)} t_i \vec{\Psi} =$$

$$= \frac{g}{2}\begin{pmatrix}\bar{\psi}^{(1)} & \bar{\psi}^{(2)}\end{pmatrix}\gamma^\mu \begin{pmatrix} B_\mu^{(3)} & B_\mu^{(1)} - iB_\mu^{(2)} \\ B_\mu^{(1)} + iB_\mu^{(2)} & -B_\mu^{(3)} \end{pmatrix}\begin{pmatrix}\psi^{(1)} \\ \psi^{(2)}\end{pmatrix} \tag{74}$$

- We obtain new Dirac equation

$$(i\gamma^\mu \partial_\mu + g\gamma^\mu B_\mu - m)\,\Psi = 0 \tag{75}$$

where B_μ is a 2×2 **matrix** $B_\mu = \sum_{i=1}^{3} B_\mu^{(i)} t_i$.

Reminder: local symmetries and gauge field

Reminder: Appearance of EM field

- How A_μ field appeared in electrodynamics? Let's look at the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \tag{76}$$

- This equation is invariant under global transformation
$$\psi(x) \to \psi'(x) = e^{i\alpha}\psi(x).$$

- All physical observables (such as **probability** $\psi^\dagger\psi$ or **current** $\bar\psi\gamma^\mu\psi$) are invariant for such a transformation

- If we demand **local gauge invariance** $\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$, the probability, current, etc. still remain invariant.

- However, the Dirac equation changes:

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\,\psi' = \\ &= (i\gamma^\mu\partial_\mu - m)\,e^{i\alpha(x)}\psi = \\ &= e^{i\alpha(x)}\,(i\gamma^\mu\partial_\mu - \gamma^\mu(\partial_\mu\alpha(x)) - m)\,\psi \quad (77)\end{aligned}$$

- How to make this theory gauge invariant?

- Let's introduce a new field. It couldn't be fermionic field, cause we can't take 3-fermion and 4-fermion interaction (3-fermion interaction doesn't even exist and both would be higher than 4-dimension operators).

- Coupling with one scalar field we also can't organize: minimal possible interaction term $g\bar{\psi}\gamma^\mu\psi\partial_\mu\phi$ has dimension 5.

- So the last possibility is a coupling with some vector field. The interaction now becomes $V = \bar{\psi}\gamma^\mu\psi A_\mu$ with some constant g. The

Dirac equation is then

$$(i\gamma^\mu\partial_\mu - m)\,\psi + g\gamma^\mu\psi A_\mu = (i\gamma^\mu D_\mu - m)\,\psi \tag{78}$$

where $D_\mu = \partial_\mu - igA_\mu$.

- Let us **along with** $\psi(x) \to \psi'(x) = e^{i\alpha(x)}\psi(x)$ also make an (**unknown so far**) transformation $A_\mu \to A'_\mu$:

$$\mathcal{L}'_F = \bar\psi\left(i\gamma^\mu\partial_\mu + \gamma^\mu\Big(gA'_\mu - \partial_\mu\alpha(x)\Big) - m\right)\psi = \\ = \bar\psi\,(i\gamma^\mu\partial_\mu + g\gamma^\mu A_\mu - m)\,\psi \tag{79}$$

- As we see, for gauge invariance of fermion field we should add vector field with transformation law $A_\mu \to A'_\mu = A_\mu + \frac{1}{g}\partial_\mu\alpha(x)$. Here our fermion field, in particular, obtain physical charge — g.

- This construction is called $U(1)$ **gauge invariance**, because $e^{i\alpha}$ is an element of $U(1)$ group.

Reminder: Appearance of EM field

- If we've measured that vector field is massless, the only possibility is to write the full theory is QED Lagrangian:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (80)$$