Solvable models for atmospheric radiative transport

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January 31, 2023

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Abstract

Two models of radiative transfer in the atmosphere are discussed: one with a uniform absorption coefficient q over the whole spectrum and one with two windows: an absorbing window with a constant absorption coefficient q and the remaining window with absorption-free radiation. The greenhouse effect is calculated as function of q and the fraction r of the absorption-free window.

1 Introduction

The effect of an increase of CO_2 in the atmosphere is a much debated issue. Some critics of the climate predictions state that there is already so much CO_2 in the atmosphere that, in the range where CO_2 absorbs, the atmosphere is inpenetrable. Indeed the absorption length for the strongest CO_2 lines is of the order of meters. More CO_2 will shorten that length a bit, but that seems hardly relevant: blacker than black is not possible. No astronomer will put an infrared telescoop on the earth: it would not observe a single star.

In this note we investigate whether more CO_2, or any other greenhouse gas, makes a difference for the radiative transport in the atmosphere. We restrict ourselves to the radiative part of the problem, since the objection formulated above concerns the flow of radiation. In this note we discuss solvable models for the radiative balance showing the basic principles. Other important mechanisms of heat flow in the atmosphere such as convection and turbulence are not discussed. By taking only the radiation into account no predictions of the actual greenhouse effect can be made. On the other hand the models clarify and quantify the influence of the radiation on the thermal profile of the atmosphere.

The earth receives energy from the sun and radiates energy back into the universe. The total amounts have to be the same, otherwise the earth would cool down or warm up. The incoming radiation is mostly in the range of frequencies of visible light and the outgoing radiation is in the thermal range with much longer wavelengths. One may think of the two streams as independent, each with their absorption coeffcients. The interaction between the radiation and the atmosphere consists of two processes (elastic) scattering and (non-elastic) absorption. Scattering shifts energy from the incoming beam to diffuse radiation, but it has no influence on the temperature distribution. For the outgowing radiation elastic scattering is of little influence, as the radiation starts already in all directions. On the other hand absorption leads to thermal re-emission which is particularly important for the outgoing radiation. The atmosphere is also a source of radiation and it gives rise to an intriguing problem of the distribution of the radiation.

There are three streams of thermal radiation. The primary stream is the radiation from the surface of the earth. While passing through the atmosphere it is partially absorbed. The absorbed energy generates secondary streams in upward and downward direction. The basic equation for the strength of these streams is the condition that at every layer in the atmosphere the net (upward) current has to be constant. From this equation for the radiation flow we derive the temperature distribution of the atmosphere and in particular the surface temperature of the earth, which gives the magnitude of the greenhouse effect.

2 The radiation input

The basic relation between temperature and black body radiation is the law of Stefan-Boltzmann

$$F = \sigma_{\rm SB} T^4 \tag{1}$$

Here T is the absolute temperature of the radiating substance, F the outgoing radiation flux and $\sigma_{\rm SB}$ is a universal constant with the value

$$\sigma_{\rm SB} = 5.670374419 * 10^{-8} {\rm W/m^2K^{-4}}. \tag{2}$$

σ involves only fundamental constants like the speed of light, the Planck constant and the Boltzmann constant.

The input radiation is determined by the surface temperature of the sun $T_\odot = 5778$ K and a number of geometrical factors. The flux of solar radiation $S_\odot$ is associated with the temperature of the sun

$$S_\odot = \sigma_{\rm SB} T_\odot^4 = 5.6704 * 10^{-8} (5778)^4 = 6.319 * 10^7 \,{\rm W/m^2}. \tag{3}$$

The radius of the sun $R_\odot = 6.957 * 10^8$ m and the distance of the earth to the sun is $R_\oplus = 1.496 * 10^{11}$ m. Thus the strength of the flux at the earth is diminuished to

$$S_\oplus = S_\odot (R_\odot / R_\oplus)^2 = 1.369 * 10^3 \,{\rm W/m^2}. \tag{4}$$

The earth has a cross-section for the radiation of πR_O^2 and a surface $4\pi R_O^2$. So on the average the earth receives the flux $F_\oplus/4$. Moreover the earth has an average albedo of 0.29, which means the 29% of the incoming flux is reflected and 71% reaches the surface of the earth. So the effective in-flux is

$$S_{\rm in} = 0.71 S_\oplus / 4 = 243 \,{\rm W/m^2}, \tag{5}$$

which heats the earth. The problem is how it is transmitted through the atmosphere back into space.

A simplistic view, due to Arrhenius, is to see the earth as a black body radiator with a uniform temperature $T_{\rm out}$. According to the Stefan-Boltzmann law $T_{\rm out}$ should be

$$T_{\rm out} = (S_{\rm in}/\sigma_{\rm SB})^{1/4} = 256 \,{\rm K}. \tag{6}$$

Compared to the average real temperature of 288 K this differs 32 K, which is attributed to the greenhouse effect.

The greenhouse effect is the temperature difference between the surface of the earth and the top of the atmosphere where the air is so dilute that thermal radiation passes without noticable absorption. There the outgoing flux must equal the flux given by

Eq. (5) in order that the earth keeps the same temperature. An alternative and more useful measure for the greenhouse effect is the ratio R of the temperature at the surface of the earth and the temperature at the top. This is a property of the composition of the atmosphere. The estimate of Arrhenius gives for the present value R=288/256=1.125. The fourth power R^4 gives the ratio of the intensity of the radiation at the surface and the top. Since the equation for the radiation is linear we may use R to estimate the ground temperature for different values of the top radiation.

An example of such different values of the outgoing radiation follows from the opposite of the Arrhenius assumption. Arrhenius assumes that the temperature on the earth is everywhere the same as it would be if the earth would be a perfect conductor. The opposite is to assume that the earth is a perfect isolator. Then there is no exchange of heat between the various lattitudes, which receive quite different amounts of radiation. Let $\phi(\theta)$ be the angle between the direction of the sun and the normal of the surface at a lattitude θ. Then we introduce the function $f(\theta)$ as

$$f(\theta) = 4\langle\cos(\phi(\theta))\rangle, \tag{7}$$

where the average is over the duration of the day and the days of the year. The 4 is for convenience since we know that the average over all lattitudes equals 1/4. Thus $f(\theta)$ gives the fraction of the radiation that enters at lattitude θ. In Fig. 1 we plot $f(\theta)$ as function of the lattitude. The differences are large. At the equator the influx is 25% larger than average and at the poles 50% smaller than average. Then we have the thermal out-flux $F_{\rm out}(\theta)$ equal to solar influx at lattitude θ

$$F_{\rm out}(\theta) = S_{\rm in}\, f(\theta). \tag{8}$$

with $S_{\rm in}$ given by Eq. (5). The same amount has to leave at the top of the atmosphere. The atmosphere has at all lattitudes the same composition. So we may use the same ratio R for all lattitudes. Thus the surface temperature at lattitude θ equals

$$T(\theta) = R\, T_{\rm out}\, f(\theta)^{1/4} = 288\, f(\theta)^{1/4}{\rm K}. \tag{9}$$

The curve $f(\theta)^{1/4}$ is also shown in Fig. 1. Although the variation in $f(\theta)$ is substantial, the 1/4 power makes the deviations from 1 much smaller. Translating them with Eq. (9) into temperatures we find 31 ^{o}C at the equator and -30 ^{o}C at the poles. In reality the differences are smaller as can be expected since heat flows from the equator to the poles leveling the temperature differences. But this temperature profile is closer to the reality than the constant value of Arrhenius.

3 Absorption and the Energy Balance

We analyze the radiation transport in a model where the earth is translation invariant in the horizontal direction, such that we have to consider only transport in the vertical direction z. The justification for this simplification is that the variations with height are much faster than those in the horizontal direction.

The primary radiation stream $I_0(z)$ comes from the surface of the earth. It starts with the value $I_0 = I_0(0)$ and decays in an exponential way due to absorption (the law of Lambert-Beer). The density of the absorbing greenhouse gases is so low that the absorption grows linearly with the density. The density $n(z)$ decreases exponentially with height z with a "thickness" $l_{\rm atm}$.

$$n(z) = n(0)\exp(-z/l_{\rm atm}). \tag{10}$$

The absorption is given by the cross-section σ times the density $n(z)$. The product gives the inverse of the absorption length $l_{\rm abs}$

$$l_{\rm abs} = \frac{1}{\sigma n(0)} \tag{11}$$

Without further source terms the radiation from the earth decays as

$$I_0(z) = I_0 \exp\left(-\sigma \int_0^z n(z')dz'\right), \tag{12}$$

Using the exponential decay of the density, the integral in the exponent can be carried out. We define the "propagator" $G_\uparrow(z,z')$ for $z > z'$

$$G_\uparrow(z,z') = \exp\left(-q[\exp(-z'/l) - \exp(-z/l)]\right), \tag{13}$$

where q is the ratio

$$q = \frac{l_{\rm atm}}{l_{\rm abs}}. \tag{14}$$

q is the dimensionless parameter controlling the radiation transport. Eq. (12) then becomes

$$I_0(z) = I_0 G_\uparrow(z,0) \tag{15}$$

The absorbed energy is re-emitted which creates secondary up- and down streams $I_\uparrow(z)$ and $I_\downarrow$. The total current $I(z)$ at layer z is the sum

$$I(z) = I_0(z) + I_\uparrow(z) + I_\downarrow(z). \tag{16}$$

The up-stream $I_\uparrow(z)$ is fed by the lower layers $z' < z$ and the down-stream by the higher layers $z' > z$

$$\left\{ \begin{array}{rcl} I_\uparrow(z) & = & \frac{1}{2}\int_0^z dz' G_\uparrow(z,z')\sigma n(z') I(z'), \\ I_\downarrow(z) & = & \frac{1}{2}\int_z^\infty dz' G_\downarrow(z,z')\sigma n(z') I(z'). \end{array} \right. \tag{17}$$

The downward decay goes with the factor

$$G_\downarrow(z,z') = \exp\left(-q[\exp(-z/l) - \exp(-z'/l)]\right), \tag{18}$$

for $z < z'$. Eqns. (17) state that the absorbed energy $\sigma n(z')I(z')$ is re-emitted half to the up-stream and half to the down-stream. This guarantees conservation of energy: all absorbed energy is re-emitted.

By inserting the Eqns. (17) into Eq. (16) one gets an integral equation for $I(z)$. Remarkably enough this impressively looking equation has an exact solution, which is derived in the Appendix A. It reads

$$I(z) = F_{\rm out}[1 + q\exp(-z/l_{\rm atm})], \qquad I_0 = F_{\rm out}(1 + q/2). \tag{19}$$

It is a straigthforward calculation to show that the form Eq. (19) indeed satisfies the equation for $I(z)$. In doing this exercise one finds for the up-stream

$$I_\uparrow(z) = F_{\rm out}\left(1 + \frac{q}{2}\right)\exp(-z/l_{\rm atm}) - I_0(z), \tag{20}$$

and for the down-stream

$$I_\downarrow(z) = F_{\rm out}\left(\frac{q}{2}\exp(-z/l_{\rm atm}\right). \tag{21}$$

Note that the upstream compensates the primary current from the earth and replaces it by a net upward stream which is larger than the downward stream.

There is another equation for the radiation which expresses the conservation of energy: the net flow of energy must stay constant as function of the height

$$F_{\rm out} = I_0(z) + I_\uparrow(z) - I_\downarrow(z). \tag{22}$$

Using the given expressions for the three ingredients one observes that this relation is indeed fulfilled.

Subtracting Eq. (16) from Eq. (22) yields yet another relation

$$I(z) = F_{\rm out} + 2I_\downarrow(z), \tag{23}$$

which is quite informative. It shows, that for $z = \infty$ where $I_\downarrow(\infty) = 0$, the radiation $I(\infty)$ equals the outflow $F_{\rm out}$ as it should.

4 Local Thermodynamic Equilibrium

Sofar the discussion concerns only the radiation, but for the greenhouse effect the temperature is important. The connection between radiation and temperature is established through the requirement of local thermodynamic equilibrium. In our case it means that the total strength of the energy stream $I(z)$ corresponds to the local temperature $T(z)$ according to

$$I(z) = \sigma_{\rm SB}T^4(z). \tag{24}$$

Another way of putting the same relation is to connect the total absorbed radiation with the thermally re-emitted radiation

$$\epsilon(z)B(z) = \sigma n(z)I(z), \tag{25}$$

where $\epsilon(z)$ is the emittance and $B(z)$ is the black-body radiation

$$B(z) = \sigma_{\rm SB}T^4(z). \tag{26}$$

Kirchhoff's law states that the emittance equals the absorptance (in local thermodynamic equilibrium)

$$\epsilon(z) = \sigma n(z). \tag{27}$$

which implies again that $I(z) = B(z)$ and thus given by Eq. (24).

For the greenhouse factor R we get the expression

$$R = \frac{T(0)}{T(\infty)} = \left(\frac{I(0)}{I(\infty)}\right)^{1/4} = (1+q)^{1/4}. \tag{28}$$

For the present $R = 1.125$ we would need $q = 0.6$.

5 The two-window model

In the previous discussed model there is a single absorption corss-section σ. In reality the absorption of the radiation is strongly frequency dependent and concentrated in absorption lines. So σ must be seen as the average absorption in the thermal frequency band. A step in the direction of a more realistic model is to consider two windows in the spectrum: one with a (uniform) absorption coefficient σ and the other window with frequencies that are not absorbed by the atmosphere.

The absorption lines are broadened by colissions which are fast with respect to the relaxation time. Then the lines get the shape of a Lorentzian. In Appendix C we outline how one can split a contribution of an absorption line into a part contributing to the absorption and a part (the wings) refering to the window without absorption.

With the introduction of two regions of different absorption a new parameter r comes into the problem. It is the size of the non-absorbing part of the spectrum. As most of the radiation is thermal the (normalized) Planck distribution $P_{\rm Pl}(\nu)$ applies. We define $P^o_{\rm Pl}(\nu)$ to be equal to the Planck distribution in the non-absorbing window and 0 outside and $P^a_{\rm Pl}(\nu)$ equal to the Planck distribution inside the absorbing window and 0 outside. The size of the windows is then given by

$$\int d\nu P^o_{\rm Pl}(\nu) = r, \qquad \int d\nu P^a_{\rm Pl}(\nu) = 1 - r. \tag{29}$$

Increase of the density of a greenhouse gas gives an increase of q and a decrease of r (the absorption lines becomes bigger and fatter).

The mathematics for the balance in the two-window model is, although similar to the single window model, substantially more involved. In this section we present the main lines and in Appendix B we give the details. In the two-window model the radiation streams are frequency dependent. We can write $I(z,\nu)$ as the sum of two streams

$$I(z,\nu) = I^o(z)P^o_{\rm Pl}(\nu) + I^a(z)P^a_{\rm Pl}(\nu), \tag{30}$$

where the upper index a refers to the absorbing window and o to the non-absorbing window. For the frequency distribution the Planck distribution applies since inside the windows the absorption is constant. $I^a(z)$ is the amplitude of the radiation that is absorbed with a cross-section σ, which amounts to

$$\int d\nu I^a(z)P^a_{\rm Pl}(\nu)\sigma n(z) = (1-r)I^a(z)\sigma n(z). \tag{31}$$

$I^a(z)$ is the key quantity, which is built-up from three components

$$I^a(z) = I^a_0 G_\uparrow(z,0) + I^a_\uparrow(z) + I^a_\downarrow(z) \tag{32}$$

It is the combination of the radiation from the surface of the earth, decaying with the factor $G_\uparrow(z,0)$ from level 0 to z, the secondary up-flow and the secondary down-flow, all in the absorbing frequency range. The up- and down streams couple to $I^a(z)$ as

$$\left\{ \begin{array}{lcl} I^a_\uparrow(z) & = & \dfrac{1-r}{2}\displaystyle\int_0^z dz' G_\uparrow(z,z')\sigma n(z') I^a(z'), \\ I^a_\downarrow(z) & = & \dfrac{1-r}{2}\displaystyle\int_z^\infty dz' G_\downarrow(z,z')\sigma n(z') I^a(z'). \end{array} \right. \tag{33}$$

Since the up- and down-stream, together with the explicit $I^a_0 G_\uparrow(z,0)$, form $I^a(z)$, we have an (integral) equation for $I^a(z)$. This equation can be converted into a differential equation in the same way as sketched in Appendix A. The result is that $I^a(z)$ is of the form

$$I^a(z) = A_+ \exp(p(z)\sqrt{r}) + A_- \exp(-p(z)\sqrt{r}). \tag{34}$$

Feeding this solution into the flux equation (22) we get three (linear) equations for I_o, A_+ and A_- from which we can determine these quantities as function of q and r. The details are given in Appendix B.

The last point is to relate the radiation intensities to the temperature. Therefore we decompose the radiation as

$$I(z,\nu) = I(z)P(z,\nu). \tag{35}$$

The distribution $P(z,\nu)$ is normalized such that

$$\int_\nu P(z,\nu) = 1, \qquad I(z) = \int_\nu I(z,\nu)d\nu. \tag{36}$$

$I(z)$ is the amplitude of the radiation at layer z. By the principle of local thermodynamic equilibrium it is related to the temperature $T(z)$ by the Stefan-Boltzmann relation

$$I(z) = \sigma_{\rm SB} T(z)^4 \tag{37}$$

On the other hand the total absorbed energy is equal to the total emitted energy, which reads with the substitution of Eq. (35)

$$\epsilon(z)B(z) = I(z)\int d\nu P(z,\nu)\sigma(\nu)n(z), \tag{38}$$

with $\epsilon(z)$ the emittance and $B(z)$ the blackbody radition. So Kirchhoff's law takes the form

$$\epsilon(z) = \int d\nu P(z,\nu)\sigma(\nu)n(z), \tag{39}$$

equating the (total) emissivity $\epsilon(z)$ with the (total) absorptance.

As in the one-window case, $I(z)$ is the sum of the three streams: the primary stream $I_0(z)$ from the earth and the induced up- and down stream $I_\uparrow(z)$ and $I_\downarrow(z)$, as in Eq. (16). By the same steps as in the one-window case we get for the greenhouse factor as in Eq. (23)

$$R^4 = \frac{I(0)}{I(\infty)} = 1 + \frac{2I_\downarrow(0)}{F_{\rm out}} \tag{40}$$

This greenhouse ratio $R(q,r)$ is shown in Fig. 2 as function of q and r. The noteworthy point is that for all $r > 0$ saturation occurs with increasing q.

For further discussion we have also plotted in Fig. 3 $R(q,r)$ as function of r for a number of values of q.

6 Conclusion

The radiative balance of the atmosphere has been discussed in a simple model with only radiative transport in the vertical direction. The greenhouse effect is represented as a ratio of the temperature at the surface of the earth and at the top of the atmosphere, where the temperature is determined by the radiative outflow which may be taken as lattitude dependent. That is more realistic than assuming, as Arrhenius, the outflow to be equal at all lattitudes.

In the simplest version of the model, absorption of the whole spectrum of frequencies are lumped into a single value. In that approximation the greenhouse factor rises with the increase of the absorption without saturation. In the next approximation the spectrum of absorption is divided into two windows: one set of frequencies that are not absorbed and one with a constant absorption coefficient. In that approximation the greenhouse factor saturates for all divisions of the the two windows (except the one with no free radiation, which is the simple version). The greenhouse factor is, however, sensitive for the size of the absorption free window.

Thus increase of the density of a greenhouse gas leads to an increase of the greenhouse factor, mainly due to the decrease of the absorption free window and less to the rise of the absorption coefficient. The Figs. 2 and 3 show that.

We have treated the atmosphere as one-dimensional, while it is three-dimensional. So we should have treated the radiation as dependent not only on the height z but also as dependent on the angle χ with the vertical direction. Thus our radiation should be considered as the angular-averaged radiation. This is an approximation as the decay due to absorption is χ dependent. The radiation from level z' to z under an angle $\chi > 0$ has a longer path than the vertical direction. So the angular-averaged radiation does not decay with the angular-averaged decay length, but the interchange of these averages seems a minor approximation.

The model can be made more realistic by introducing more different windows of absorption, but the mathematical evaluation which is exact in the two discussed versions, soon becomes too complicated to be useful.

Of course the real greenhouse effect is influenced by other factors such as convection, both in the vertical and horizontal direction. The justification to consider only the radiative processes is amongst others that radiative flow is fast with respect to other equalization processes.

A The Solution

We rewrite Eq. (24) by scaling out $F_{\rm out}$ and making some substitutions

$$I_0 = F_{\rm out}\, i_0, \qquad B(z) = F_{\rm out}\, b(p), \qquad p(z) = q\exp(-z/l). \tag{41}$$

The variable p runs between 0 (top of the atmosphere) and q (surface of the earth). In these variables the propagator $G(z,z')$ becomes

$$G(z,z') = \exp(p-p'), \qquad \text{and} \qquad dp = -(q/l)\exp(-z/l)dz. \tag{42}$$

Then Eq. (24) becomes in the new variables

$$1 = i_0\exp(p-q) + \frac{1}{2}\left(\int_p^q \exp(p-p') - \int_0^p \exp(p'-p)\right) b(p')dp'. \tag{43}$$

The first step is to multiply the equation with $\exp(-p)$

$$\exp(-p) = i_0\exp(-q) + \frac{1}{2}\left(\int_p^q \exp(-p') - \int_0^p \exp(p'-2p)\right) b(p')dp' \tag{44}$$

and then differentiate this equation with respect to p

$$-\exp(-p) = -\frac{1}{2}[\exp(-p)+\exp(-p)]b(p) + \int_0^p \exp(p'-2p)dp' b(p'). \tag{45}$$

Note that the flux of the surface of the earth has disappeared from this equation, which gives also the boundary condition for $p = 0$

$$b(0) = 1. \tag{46}$$

Now multiply with $-\exp(2p)$

$$\exp(p) = \exp(p)b(p) - \int_0^p dp' \exp(p')b(p') \tag{47}$$

and differentiate again

$$\exp(p) = \exp(p)b(p) + \exp(p)\frac{db(p)}{dp} - \exp(p)b(p), \tag{48}$$

which gives the derivative of $b(p)$ as

$$\frac{db(p)}{dp} = 1. \tag{49}$$

Together with the boundary condition Eq. (46) we find

$$b(p) = 1 + p = i(p). \tag{50}$$

With this solution we get for the up- and down current

$$i_\uparrow(p) = 1 + \frac{p}{2} - \left(1 + \frac{q}{2}\right)\exp(p - q), \qquad i_\downarrow(p) = \frac{p}{2}. \tag{51}$$

Note that with i_0 equal to

$$i_0 = 1 + \frac{q}{2}, \tag{52}$$

both the flux equation (24) as well as the relation

$$i(p) = i_0 \exp(p - q) + i_\uparrow(p) + i_\downarrow(p) \tag{53}$$

is fulfilled.

B Details of the two-window model

In this appendix we give the details the solution of the two-window model. As in the single window model we scale out the magnitude of the outgoing flux F_{out} (see Eq. (41). In general lower case symbols are scaled quantities. That holds also for the coefficients $A_\pm$ of the exponentials

$$A_\pm = F_{\text{out}}\, a_\pm \tag{54}$$

We use, instead of the height variable z, the variable p running between $0 \le p \le q$. The scaled form the Eqns. (33) then reads

$$\left\{\begin{array}{rcl} i^a_\uparrow(p) & = & \dfrac{1-r}{2}\displaystyle\int_p^q dp' \exp(p - p')\, i^a(p'), \\ i^a_\downarrow(p) & = & \dfrac{1-r}{2}\displaystyle\int_0^p dp' \exp(p' - p)\, i^a(p'). \end{array}\right. \tag{55}$$

The third component of $i^a(p)$, the flow from the ground level in the absorbing window, reads explicitly

$$i_0^a(p) = i_0 \exp(p - q). \tag{56}$$

The equation for $i^a(p)$ results from summing the two Eqs. (55) and Eq. (56)

$$i^a(p) = i_0 \exp(p - q) + \frac{1 - r}{2} \left[\int_p^q dp' \exp(p - p')\, i^a(p') + \int_0^p dp' \exp(p' - p)\, i^a(p') \right]. \tag{57}$$

This equation can be turned into a differential equation by applying the operator

$$\mathcal{O}(p) = e^{-p} \frac{d}{dp} e^{2p} \frac{d}{dp} e^{-p} \tag{58}$$

to the equation with the result

$$\frac{d^2 i^a(p)}{dp^2} = r\, i^a(p). \tag{59}$$

The solution is therefore of the form

$$i^a(p) = a_+ \exp(\sqrt{r}\, p) + a_- \exp(-\sqrt{r}\, p), \tag{60}$$

with $a_\pm$ as adaptable constants. Inserting the solution into Eq. (57) we get a relation between a_+ and a_-

$$a_+(1 - \sqrt{r}) + a_-(1 + \sqrt{r}) = 0 \tag{61}$$

and an expression for i_0

$$i_0 = a_+ \frac{1 + \sqrt{r}}{2} \exp(q\sqrt{r}) + a_- \frac{1 - \sqrt{r}}{2} \exp(-q\sqrt{r}). \tag{62}$$

The two Eqns. (61) and (18) give a_+ and a_- as function of i_0.

B.1 The Flux Equation

The relation between I_0 and $F_{\rm out}$ follows from the flux equation. At every layer the nett flux upward has to be constant and equal to the flux $F_{\rm out}$ at the top of the atmosphere. In scaled form the flux equation reads

$$1 = i_0(p) + i_\uparrow(p) - i_\downarrow(p). \tag{63}$$

In the scaled variables this is a relation for i_0. These fluxes are integrated over the frequencies. So one has

$$i_0(p) = r\, i_0^o(p) + (1 - r)\, i_0^a(p) = r\, i_0 + (1 - r)\, i_0 \exp(p - q). \tag{64}$$

For the other fluxes we have similarly

$$\left\{ \begin{array}{lcl} i_\uparrow(p) & = & r i_\uparrow^o(p) + (1 - r)\, i_\uparrow^a(p), \\ i_\downarrow(p) & = & r i_\downarrow^o(p) + (1 - r)\, i_\downarrow^a(p). \end{array} \right. \tag{65}$$

The streams $i_\uparrow^a(p)$ and $i_\downarrow^a(p)$ are given in Eq. (55) in terms of $i^a(p)$. The streams $i_\uparrow^o(p)$ and $i_\downarrow^o(p)$ read

$$i_\uparrow^o(p) = \frac{1 - r}{2} \int_p^q dp'\, i^a(p'), \qquad i_\downarrow^o(p) = \frac{1 - r}{2} \int_0^p dp'\, i^a(p'), \tag{66}$$

So the flux equation Eq. (63) is another equation for $i^a(p)$. It is rewarding that the solution Eq. (60) also satisfies this equation and that the same expression Eq. (62) for i_0 applies. In addition one more relation between the $a_\pm$ follows from the presence of the left hand side of flux equation. With that relation and Eq .(61) we get the values of $a_\pm$

$$a_+ = \frac{2(1+\sqrt{r})}{\det(q,r)}, \qquad a_- = -\frac{2(1-\sqrt{r})}{\det(q,r)}, \tag{67}$$

with

$$\det(q,r) = \sqrt{r}\left[\left(\frac{\exp(q\sqrt{r})}{1-\sqrt{r}}+1\right)(1+\sqrt{r}) + \left(\frac{\exp(-q\sqrt{r})}{1+\sqrt{r}}+1\right)(1-\sqrt{r})\right]. \tag{68}$$

Inserting the expression (68) for $a_\pm$ into Eq. (62) gives the value of i_0 in terms of q and r. This completes the solution of the flux equation.

B.2 The Greenhouse Factor

For the computation of the greenhouse factor we follow section 5. Eq. (16) for the total stream reads in scaled form

$$i(p) = i_0(p) + i_\uparrow(p) + i_\downarrow(p). \tag{69}$$

Using the flux equation (63) it can be written as

$$i(p) = 1 + 2i_\downarrow(p). \tag{70}$$

$i_\downarrow(p)$ has two components

$$i_\downarrow(p) = i_\downarrow^o(p) + i_\downarrow^a(p). \tag{71}$$

The first component is related to $i^a(p)$ in Eq. (66) and the second component similarly in Eq. (55). Using the form Eq. (60) for $i^a(p)$ we find

$$2i_\downarrow(p) = a_+(\exp(p\sqrt{r})-1) + a_-(\exp(-p\sqrt{r})-1) \tag{72}$$

The value $i(0) = 1$ corresponds to the radiation at the top and $i(q)$ to the radiation at the surface. So we find for the greenhouse ratio

$$\frac{i(q)}{i(0)} = 1 + 2i_\downarrow(q). \tag{73}$$

Using the expression for $i_\downarrow(q)$ we find

$$R^4(q,r) = 1 + a_+(\exp(q\sqrt{r})-1) + a_-(\exp(-q\sqrt{r})-1). \tag{74}$$

C A single absorption line

Absorpton lines, broadened bij collisions, have a Lorentzian shape, $q(\nu) = qS(\nu)$, with a (normalized) distribution $S(\nu)$

$$S(\nu) = \frac{1}{\pi}\frac{\Delta}{(\nu-\nu_0)^2+\Delta^2}, \tag{75}$$

where ν_0 is the center of the line and Δ the width. The absorbing part of the line is the sector δ in the center, ranging from ν_- to ν_+, symmetric around the center ν_0 of the line

$$\nu_+ = \nu_0 + \delta/2, \qquad \nu_- = \nu_0 - \delta/2. \tag{76}$$

The criterion is that the integrated wings beyond $\nu_\pm$ do not exceed 1

$$1 = 2\int_{\nu_+}^{\infty} d\nu\, q(\nu). \tag{77}$$

Inserting the Eqn. (75) we find

$$1 = \frac{2q}{\pi}\int_{\nu_+}^{\infty} d\nu \frac{\Delta}{(\nu-\nu_0)^2+\Delta^2} = \frac{2q}{\pi}\left(\frac{\pi}{2} - \arctan\frac{\delta}{2\Delta}\right). \tag{78}$$

This gives for δ the expression Eq. (32).

The fraction $1-r$ of the absorbing window is proportional to δ, which covers the inner part of the line. If we double the density of the greenhouse gas the value of q will double. With Eq. (32) we can calculate the effect on δ and therefore on $1-r$. Solving for δ yields becomes

$$\delta = 2\Delta\tan\left(\frac{\pi}{2} - \frac{\pi}{2q}\right) \tag{79}$$

For large q this becomes

$$\delta \simeq \frac{4q\Delta}{\pi} \tag{80}$$

Thus doubling of q means that $1-r$ increases with a factor 2 or that the new r' equals

$$r' = 2r - 1. \tag{81}$$

This holds for $r > 1/2$ and under the assumption that the spectral lines do not merge. When the absorption lines are equidistant and $r = 1/2$ the free space between the absorbing regions equals the absorbing range. So doubling the absorbing regions closes the gaps between the absorbing regions.

In Fig. 3 we show the greenhouse factor $R(q,r)$ as function of r for a number of values of q. For the situation with little greenhouse effect ($r \simeq 1$) we see that doubling has little effect on r, but when r gets smaller the change is r is much more important than the doubling of q.

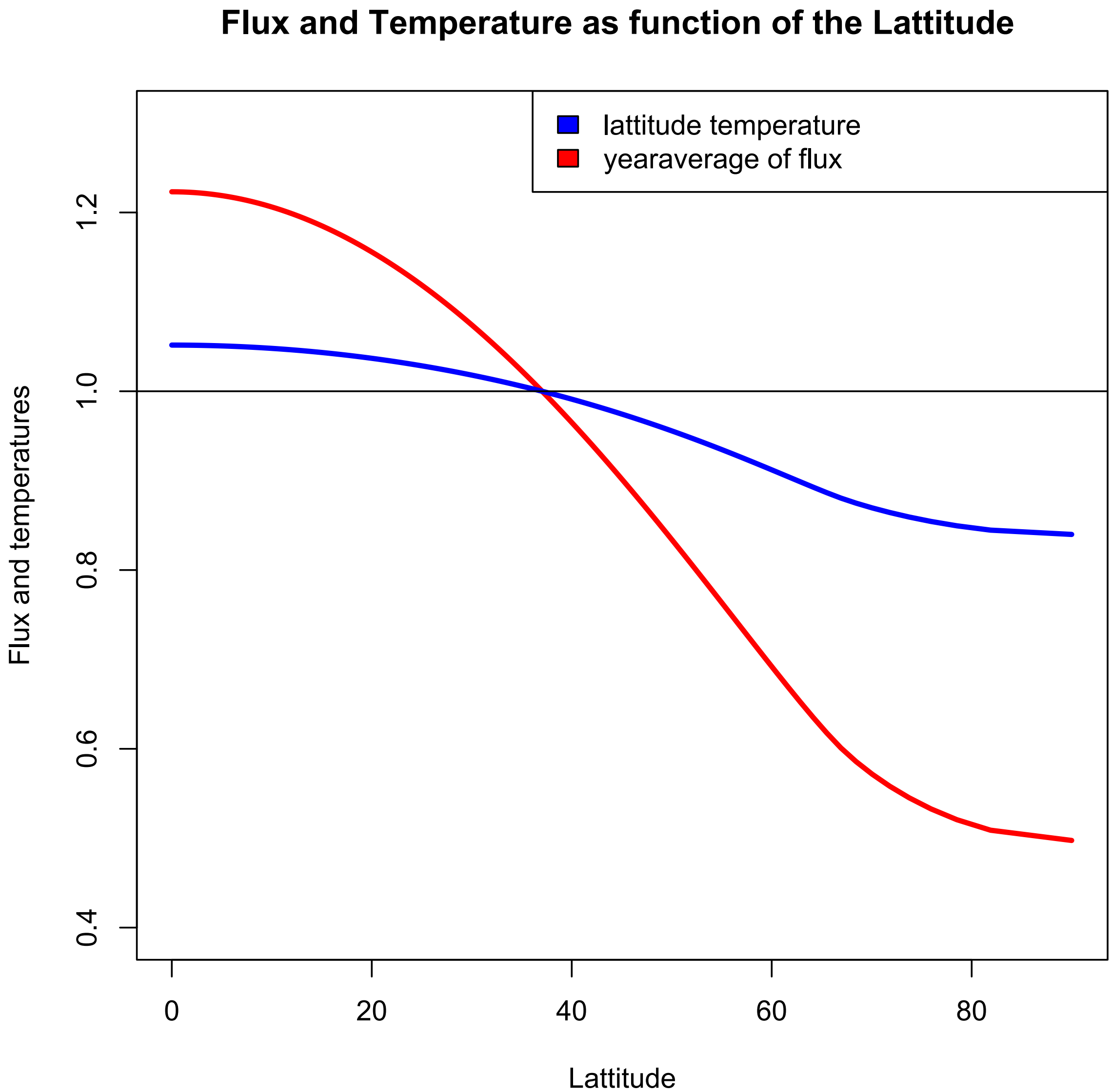

Figure 1: Yearly average of the radiation and the resulting ground temperatures R as function of the lattitude θ

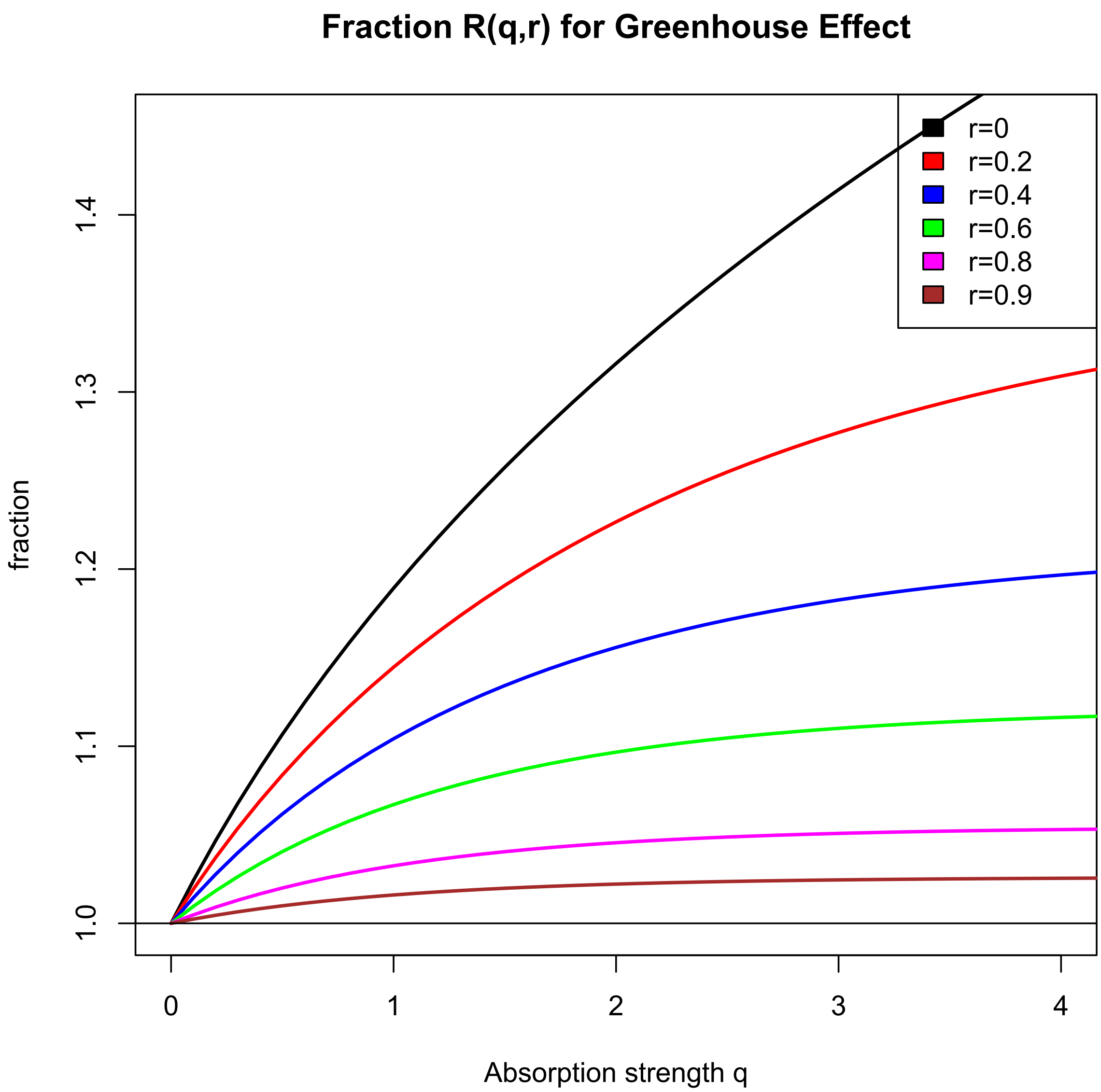

Figure 2: The Greenhouse Effect $R(q, r)$ as function of the absorption coefficient q and the window size r.

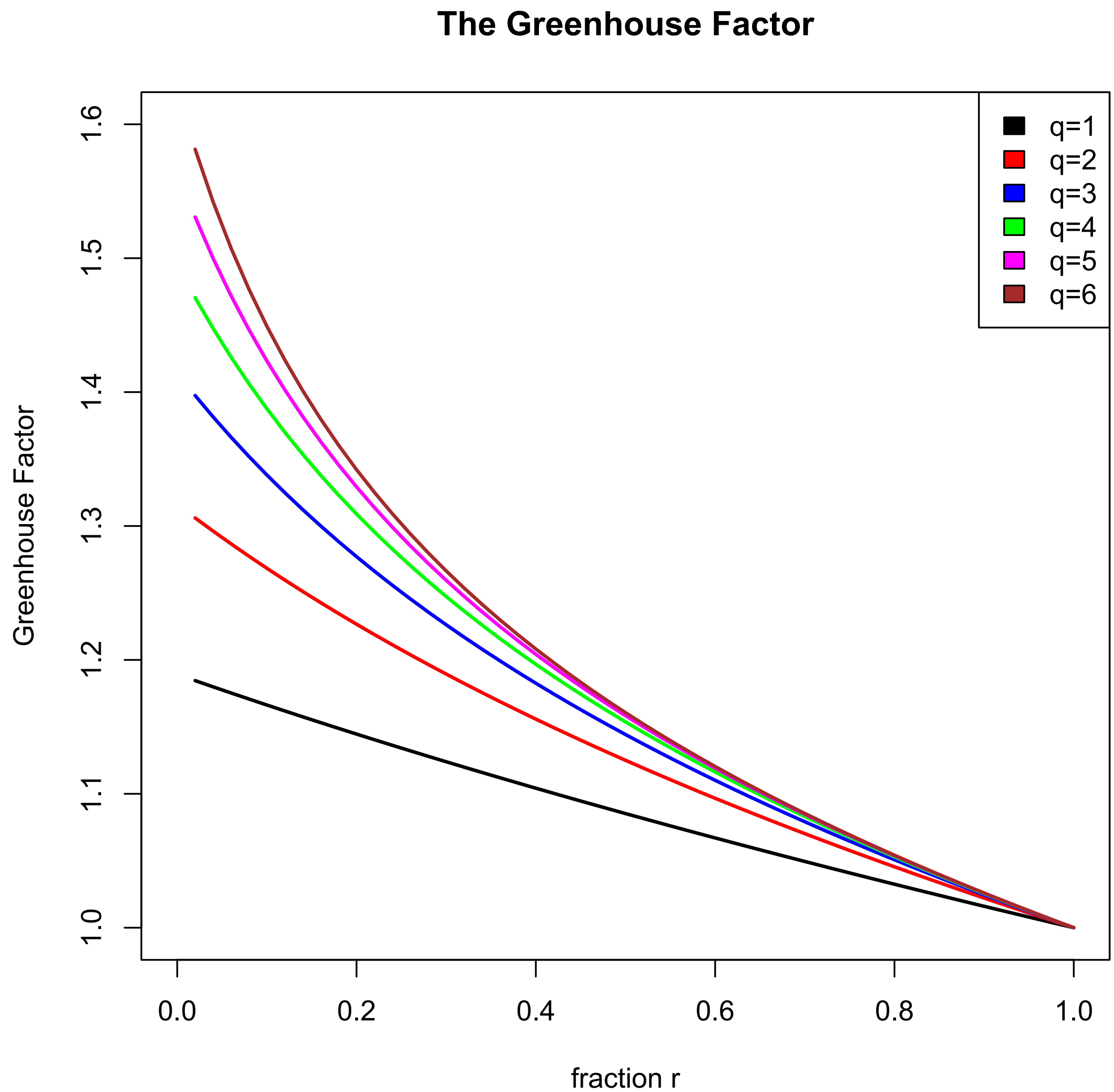

Figure 3: The Greenhouse Factor $R(q, r)$ as function of the window size r for a number of absorption coefficients q.