The standard model of finance

Merton-Black-Scholes model for option pricing

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With inspiration from:
J. Tinbergen, T.C. Koopmans, E. Majorana, F.L.J. Vos

Outline

I. Introduction: heat conduction and diffusion
II. Brownian motions and stochastic calculus
III. The standard model of finance
a. Preliminaries; portfolio dynamics and arbitrage
b. Merton-Black-Scholes model for option pricing
c. The Black-Scholes formulas
IV. (Implied) volatility; universality?

R.N. Mantegna and H.E. Stanley, *An introduction to Econophysics: Correlations and Complexity in Finance* (2000, 2007)

J.-P. Bouchaud and M. Potters, *Theory of Financial Risk and Derivative Pricing: From Statistical Physics to Risk Management* (1997, 2009)

J. Voit, *The Statistical Mechanics of Financial Markets* (2000, 2005)

Introduction: heat conduction and diffusion

Physical diffusion vs. stochastic diffusion

Fourier 1807 heat equation $T(t, x, y, z)$
Laplace 1809 stochastic diffusion equation $P(x, n)$
Einstein 1905 synthesis via Brownian motion ($\rightarrow$ estimate of Avogadro's number)

$$\frac{\partial P}{\partial t} = D_s \frac{\partial^2 P}{\partial x^2} \qquad \langle x^2 \rangle = 2D_s t$$

Bachelier 1900 (!) *Théorie de la spéculation* $n \rightarrow$ time
Mandelbrot 1963 cotton prices

Brownian motion

Random walk: every time interval Δt take step ℓ left or right; what is position $x = n\ell$ after time $t = N\Delta t$?

Brownian motion: stochastic process resulting from taking the random walk to the continuous limit. Solution:

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{(x-m)^2}{2\sigma^2 t}}$$

$$\langle x^2 \rangle = \sigma^2 t \quad \text{linear in } t; \quad D_s \equiv \tfrac{1}{2}\sigma^2$$

$$dX = \sigma dW \quad \text{where} \quad dW = \mathcal{O}\left(\sqrt{dt}\right) \text{ (Wiener process)}$$

Brownian motion with drift:

$$dX = \mu dt + \sigma dW$$

Geometric Brownian motion:

$$dS = \mu S dt + \sigma S dW$$

Using the Itô formula from stochastic calculus:

$$Z = \ln S \quad \longrightarrow \quad dZ = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW$$

Log-normal distribution: (initial value $S(t_0) = S_0, \quad \tau = t - t_0$)

$$P(S, t; S_0, t_0) = \frac{1}{S\sqrt{2\pi\sigma^2\tau}} \exp\left\{ -\frac{\left[\ln\left(\frac{S}{S_0}\right) - (\mu - \frac{1}{2}\sigma^2)\tau\right]^2}{2\sigma^2\tau} \right\}$$

Stochastic calculus

Underlying process: $W(t)$
Primary process: $X(t)$ $\qquad dX = a(X,t)dt + b(X,t)dW$
Derived process: $Z(t) = F(t, X(t))$
Itô formula:

$$dF = \left[\frac{\partial F}{\partial t} + a(X,t)\frac{\partial F}{\partial x} + \tfrac{1}{2}b^2(X,t)\frac{\partial^2 F}{\partial x^2}\right] dt + b(X,t)\frac{\partial F}{\partial x} dW$$

fluctuating part of the primary process $X(t)$ contributes to the drift of the derived process $Z(t)$!

$(\quad X \equiv S \;, \quad a \equiv \mu S \;, \quad b \equiv \sigma S \;, \quad F \equiv \ln S \quad)$

Preliminaries

(Partly in) Lecture I: Financial markets

A. Changes in stock prices are log-normally distributed
$\longrightarrow$ geometric Brownian motion

B. For a forward contract the enforced arbitrage price is:
$F = S_0\, e^{rT}$ (= forward payment written in contract;
S_0: price at time of contract, r: interest rate, T: maturity)
Independent of distribution of stock prices (μ, σ),
i.e. independent of fluctuations of underlying stock !

A forward contract is a contract between two parties on the delivery of an asset at a certain time T in the future at a certain price. The contract is binding to **both** parties.

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C. European call option: a contract that gives the holder the right (but not the obligation) to buy the underlying asset for the (strike) price K at (maturity) time T (K, T specified in the contract).

D. No-arbitrage principle: *In a market free of arbitrage, any riskless portfolio must yield the risk-free interest rate r.*

Portfolio dynamics and arbitrage

Value of portfolio:

$$V_{\mathbf{x}} = \mathbf{x} \cdot \mathbf{S} = x_0 B + x_1 S$$

Self-financing portfolio:

$$dV_{\mathbf{x}} = \mathbf{x} \cdot d\mathbf{S} \qquad (t \geq 0)$$

An arbitrage is a portfolio for whose value holds:

(i) $V(0) = 0$ — start with nothing

(ii) $V(t) \geq 0$ with probability 1 for all $t > 0$ — cannot loose money

(iii) $V(T) > 0$ with positive probability for some $T > 0$ — chance of profit

Chance of a riskless profit out of nothing!

Merton-Black-Scholes model for option pricing

Two main assumptions:

(i)
- $dB = rBdt$ (*interest upon interest*; r: constant)
- $dS = \mu Sdt + \sigma SdW$ (*geometric Brownian motion*)

(ii) The market is free of arbitrage

⋆ (many) additional practical/technical assumptions

F. Black and M. Scholes, J. Polit. Econ. **81**, 637 (1973)
R. C. Merton, Bell J. Econ. Manag. Sci. **4**, 141 (1973)

Merton and Scholes, Nobel Prize in Economics 1997

Additional assumptions in MBS model

(i) trading of assets is **continuous**

(ii) selling of assets is possible at any time

(iii) there are **no transaction costs**

(iv) all market participants can lend and borrow money at the **same, constant interest rate** r

(v) there are no dividend payments between $t = 0$ and $t = T$.

(vi) ... (taxes, short selling, ...)

Idealized financial markets

European call option: $C(S, t; K, T)$

$$dC = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 C}{\partial S^2}\right] dt + \sigma S \frac{\partial C}{\partial S} dW$$

Delta-hedging portfolio: $\Pi(t) = C(S, t) - \Delta S$

$$d\Pi = \left[\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 C}{\partial S^2} - \mu \Delta S\right] dt + \sigma S \left(\frac{\partial C}{\partial S} - \Delta\right) dW$$

Eliminate the risk (i.e. the stochastic term): $\Delta = \frac{\partial C}{\partial S}$!!

The *No-arbitrage principle*, i.e. $d\Pi = r\Pi dt$, now leads to the Black-Scholes differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0$$

Boundary condition: $C(S, T) = \max(S - K, 0)$

Solution to BS equations: change of variables: $t, S \to \tau, x$

$$\tau = \frac{T-t}{(2/\sigma^2)} \qquad x = \ln\left(\frac{S}{K}\right)$$

$$u(x,\tau) = e^{\alpha x + \beta^2 \tau} \frac{C(S,t)}{K}$$

With appropriate choice of α and β: $\alpha = \frac{1}{2}\left(\frac{2r}{\sigma^2} - 1\right)$ $\beta = \frac{1}{2}\left(\frac{2r}{\sigma^2} + 1\right)$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

heat/diffusion equation!

Terminal condition at $t = T$ becomes initial condition at $\tau = 0$:

$$u(x, \tau = 0) = \begin{cases} 0 & \text{for } x < 0 \\ e^{\beta x} - e^{\alpha x} & \text{for } x \geq 0 \end{cases}$$

Using the Green function $G(x, x'; \tau)$ of the heat equation:

$$G(x, x'; \tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-(x-x')^2/4\tau} \quad ,$$

$$u(x, \tau) = \int_{-\infty}^{\infty} G(x, x'; \tau)\, u(x', \tau = 0)\, dx' = I(\beta) - I(\alpha)$$

with $I(a) = e^{a^2\tau + ax} N\left(\frac{x + 2a\tau}{\sqrt{2\tau}}\right)$ and $N(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$

Going back to original variables S and t leads to the Black-Scholes formulas:

The Black-Scholes formulas

$$C(S,t) = S\,N(d_1) - K\,e^{-r(T-t)}\,N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

stock price S strike price K

risk free interest rate r maturity T

$$N(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds$$

volatility σ !!

Discussion

Main idea of MBS: Riskless portfolio, consisting of option and underlying asset, is possible. The stochastic process (i.e. the risk) can be eliminated since both stock and option depend on the same source of uncertainty!

Important achievement of MBS:

1. In *idealized markets*, the risk associated with an option can be hedged away completely (Δ-hedging).
2. The writer (seller) of an option does not need to ask for a risk premium (because μ, the average rate of return of the stock, has dropped out of the equations).

Discussion

The Black-Scholes differential equation is similar to the Fokker-Planck and Schrödinger equations of **Physics** and the Kolmogorov equation of **Mathematics** (but with important differences!).

The Merton-Black-Scholes theory contains important aspects of both Probability Theory/Statistics (probability distributions, stochastic processes) and Game Theory (arbitrage, strategy, decisions). The model allows for an exact, non-trivial solution. This may be compared to e.g. the Ising model in two dimensions for magnetism in (statistical) physics.

Less appropriate would be to compare with the *Standard Model of Elementary Particle Physics*

Implied volatility: σ_{imp}

$$C_{\text{market}}(S, t; r, \sigma; K, T; \ldots) \equiv C_{\text{BS}}(S, t; r, \sigma_{\text{imp}}; K, T)$$

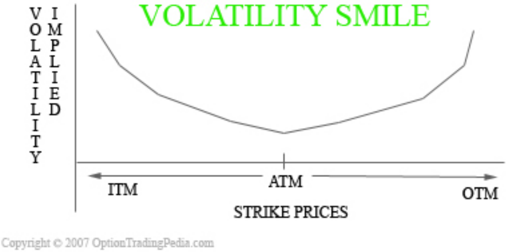

Figure: Implied volatility of options on the same underlying asset and expiring on the same day as a function of strike prices.
ITM: in-the-money; ATM: at-the-money; OTM: out-of-the-money

VAEX-index: Volatility index voor opties op aandelen genoteerd in de AEX, voor de periode van 21 augustus 2008 tot en met 20 augustus 2009.

VIX-index: Volatility index voor opties aan de CBOE (Chicago Board Options Exchange), gebaseerd op aandelen in de S&P500 (Standard & Poor's 500).

Figure: Volatility indices VAEX (options on AEX stocks) and VIX (options on S&P500 stocks), August 2008 - August 2009.

A case of universality?

Inspirators

J. Tinbergen (1903-1994) Ph.D. 1929 Physics, Leiden; Ehrenfest
(1st) Nobel Prize Economics 1969

T.C. Koopmans (1910-1985) Ph.D. 1936 Physics, Leiden; Kramers
Koopmans' theorem (QM) 1934; Nobel Prize Economics 1975

E. Majorana (1906-"1938") child prodigy of Italian physics
paper 1936/1942: *Il valore leggi statistiche nella fisica e nelle scienze sociali*

The Great Depression (US: 1929 - 1933/1941; NL: 1929 - 1936)