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NON-LINEAR HYDRODYNAMIC FLUCTUATIONS AROUND EQUILIBRIUM

Remarks in connection with an article by R.F. Fox

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In reply to an appendix in our article[1]) on the above subject, Fox wrote an interesting note[2]) which induces us to make the following remarks:

i) Fox admits that his original criticism[3,4,5]) of non-linear hydrodynamic fluctuation theory was incorrect ("flawed"). This criticism was based on the fact that the average of the (fluctuating analogue of the) Rayleigh dissipation function, i.e. the term $(P_{\alpha\beta} - p\delta_{\alpha\beta} + \tilde{S}_{\alpha\beta})D_{\alpha\beta}$ in Fox's notation, and therefore the time derivative of the mean temperature seemed to diverge. As we showed, this was due to the fact that Fox did not correctly retain all terms in the evaluation of this average. In fact, we showed on general grounds that this average is zero and verified in a special case that the divergencies indeed cancel.

ii) Fox, in his reply to our article, now analyses the autocorrelation function of the temperature which contains the second moment of the above function. He then finds that this second moment diverges. Using translational invariance, the equal time temperature autocorrelation function may be written in the following form

$$\langle \Delta T(\boldsymbol{k}, t)\Delta T(\boldsymbol{k}', t)\rangle_{\text{eq}} = (2k_{\text{B}}T^2_{\text{eq}}/C_v\rho_{\text{eq}})(1 + f(\boldsymbol{k}))(2\pi)^3\delta(\boldsymbol{k} + \boldsymbol{k}').$$

For notations we refer to refs. 1 and 2. From Fox's eq. (41) one obtains the contribution to $f(\boldsymbol{k})$ due to the second moment of the fluctuating analogon of the Rayleigh dissipation function. This contribution diverges.*

In our article we showed that the Einstein equilibrium distribution function[6]) is a stationary solution of the Fokker–Planck equation for the distribution function of non-linear hydrodynamic fluctuations.** Every distribution function will

*This is the case notwithstanding a minor algebraic error in his eq. (41). One should replace $(1 - 3\cos^2\alpha + 4\cos^4\alpha)$ by $(1 + \cos^2\alpha)$ in his formula.

**We restricted our discussion to the case where the dissipative fluxes are linear in the gradients while the so-called Onsager coefficients are constants. All non-linearities due to convective terms and the equation of state were taken into account.

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approach in the course of time this equilibrium distribution function.† The equal time–temperature autocorrelation function can therefore be obtained from the Einstein equilibrium distribution function according to our scheme, as obviously it should. This was one of the features we set out to establish. Fox's function $f(\boldsymbol{k})$ in the above equation must therefore also follow if one knows the entropy of the system as a function of the hydrodynamic fields. One is then faced with the difficulty that the dependence of the entropy of the system on the Fourier components of the hydrodynamic fields for molecular values of the wave vector is very different from the dependence for hydrodynamic values *and in fact not even properly defined.* To cope with this difficulty it is necessary to introduce cells which are sufficiently small but larger than the molecular size. This is the procedure followed in our paper. Alternatively one may introduce a cut-off wave number k_c of the order of an inverse molecular length. As Fox correctly points out, $f(\boldsymbol{k})$ may in principle approach infinity in the limit $k_c \to \infty$.

Of course, if one calculates the function $f(\boldsymbol{k})$ or $f(\boldsymbol{k}, \omega)$ for the more general case ($f(\boldsymbol{k}) = (1/2\pi) \int d\omega\, f(\boldsymbol{k}, \omega)$), one must in principle, as we have shown in our paper, take into account *all non-linear terms* in the hydrodynamic equations of motion in order to be consistent. Fox, in his analysis, only takes the fluctuating analogon of the Rayleigh dissipation term into account and neglects convection terms and the non-linearity due to the equation of state. Such a procedure is not necessarily consistent.

iii) It is a well-known aspect of continuum theories (to which we also refer in our paper) that a wave vector cut-off is often needed to tame divergencies, in particular in statistical considerations. An example is the cut-off introduced in Debye's theory of the specific heat. Also in mode–mode coupling theories such cut-offs are a common feature. Furthermore, in the field-theoretical renormalization group theories a cut-off is used and is in fact crucial in the determination of the renormalization group equations. More examples may easily be given. Even in linear hydrodynamic fluctuation theory a quantity such as the autocorrelation function of the kinetic energy density $\frac{1}{4}\rho_{eq}^2\langle|\boldsymbol{v}(\boldsymbol{r}, t)|^2|\boldsymbol{v}(\boldsymbol{r}', t)|^2\rangle_{eq}$ may easily be shown to diverge (a cut-off wavevector would give a finite result). Fox, however, in his concluding remarks states that "No cut-off is required in the usual [i.e. linear] equilibrium theory, even though it is a continuum theory. Many experimentally compatible results have been calculated in the continuum limit which is not inherently divergent". It is true that in the linear theory the usual problems to be solved, such as e.g. the form of the light scattering spectrum, do not require the use of a cut-off. The fact that one needs a cut-off to calculate the autocorrelation function of the kinetic energy

†Using the so-called potential conditions, eqs. (5.2) and (5.3) of our paper, one may easily verify that the usual H function $\int P(\{\underline{\alpha}_n\})\ln(P(\{\underline{\alpha}_n\})/P^{eq}(\{\underline{\alpha}_n\})\Pi_n\, d\underline{\alpha}_n$ is a Liapunov function.

density in the linear theory should certainly not be construed to imply that this theory is inconsistent and useless. Both, in the linear as well as in the non-linear theory, it depends on the problem under consideration, and in particular on the relative importance of small distances, whether a cut-off is needed or not.

iv) We certainly agree that "while the time reversal argument proposed is valid for equilibrium, it is not applicable away from equilibrium in hydrodynamics". As is already clear from the title of our paper[1]), as well as the foot note in its introduction, we only considered *fluctuations around equilibrium*. It seemed unnecessary to stress in this connection the obvious fact that microscopic reversibility or detailed balance does not hold as a general property for stationary non-equilibrium states. Pertinent remarks on this point can be found in a paper by Landauer.[7])

In conclusion we may state that, while wave vector cut-offs are frequently needed in continuum theories (and not only because of the occurrence of multiplicative noise terms), this certainly does not invalidate the corresponding theories.

References

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4) R.F. Fox, Supp. Progr. Theor. Phys. **64** (1978) 425.
5) R.F. Fox, Phys. Rep. **48C** (1978) 179.
6) See e.g. L.D. Landau and E.M. Lifshitz, Course of Theoretical Physics, vol. 5 (Pergamon, New York, 1980).
7) R. Landauer, Physics Today, November (1978) 23.