Physica **112A** (1982) 65–100 *North-Holland Publishing Co.*

EXACT DIFFERENTIAL RENORMALIZATION GROUP EQUATIONS FOR ISING MODELS ON SQUARE LATTICES

Wim VAN SAARLOOS

Instituut - Lorentz, Rijksuniversiteit te Leiden, Nieuwsteeg 18, 2311 SB Leiden, The Netherlands

Received 29 September 1981

The differential real space renormalization theory of Hilhorst et al. is applied to Ising models on square lattices with nearest-neighbour interactions only. The renormalization flow equations for these two interaction parameters contain two auxiliary parameters; these parameters have to be determined by solving two additional equations, one partial differential equation and one ordinary equation. The necessity of introducing these additional parameters is explained by arguing that for the present formulation of differential real space renormalization theory in d dimensions, at least $d + 1$ parameters are required. The concept of local fixed point is introduced; this fixed point can be determined by solving *algebraic* equations. The linearized flow around it describes local properties of the system and is therefore related to the critical properties of homogeneous Ising systems. We study temperature-like perturbations around the local fixed point and find a unique eigenvalue $y_T = 1$, in agreement with the known exact result.

1. Introduction

The introduction by Hilhorst, Schick and Van Leeuwen[1,2]) (to be referred to as HSL) of a scheme that yields exact renormalization equations in differential form for the two-dimensional Ising model, has had important methodological value. For, while the mere existence of such equations is already surprising, the differential real space renormalization scheme of HSL has provided a means to check the ideas of renormalization theory for the most celebrated model in the theory of critical phenomena.

The ideas of HSL have in recent years been extended in various directions. Most importantly, the renormalization group (RG) equations for the Ising model have been analysed more thoroughly by several workers. Knops and Hilhorst[3]) have studied these equations in the critical subspace. In addition to an exact solution of the nonlinear flow equations in this subspace, they also found a new class of fixed-point solutions. Their solution of the flow equations was recently rewritten in a simple closed form by Ray[4]). Payandeh and Van Leeuwen[5]) have set up a calculational scheme to obtain successive approximations for the magnetic critical exponent, and Hilhorst and Van Leeuwen[6]) used the original transformation of HSL as a starting point for a perturbation expansion that yields information about the critical behaviour of the Ashkin–Teller model. The applicability of differential real space renormalization theory to other lattice models has proved to be possible too: Van Leeuwen[7]) showed that the q-state Potts model may be analysed in the $q \to 0$ limit, Yamazaki et al.[8–10]) studied the d-dimensional Gaussian model, and van Saarloos et al.[11]) the one-dimensional Ising chain in a magnetic field or with quenched random interactions, while Dekeyser and Stella[12,13]) adapted the theory so as to analyse both the equilibrium and the dynamical properties of the Van der Waals spin system.

For the construction of their RG equations HSL considered the Ising model on a triangular lattice in the absence of a magnetic field. It is inherent to differential real space renormalization theory that spatially varying interactions should be considered. Consequently, HSL had to take the three different nearest-neighbour interactions of the triangular lattice into account. In a way, their equations then contain a redundant variable, since it is well known that a general Ising model on a triangular lattice is thermodynamically equivalent (in a sense to be specified later) to an Ising model on a square lattice, which has just two interaction parameters. The question therefore arises whether it would be possible to formulate a renormalization transformation directly in terms of the two interaction parameters of a square lattice. In this paper we will indeed derive exact differential RG equations for the Ising model on a square lattice, without achieving however the reduction in the number of parameters. In fact, we will argue that it is an artifact of the presently known formulation of differential real space renormalization theory that in d dimensions at least $d+1$ parameters are needed in order to study the critical properties pertaining to homogeneous systems. This, of course, excludes the possibility to formulate differential RG equations for square Ising models in terms of the two nearest-neighbour interactions alone.

Our RG equations are derived in section 2 on the basis of a restructuring transformation by Hilhorst[14]) for square Ising models. The restructuring transformation is based on the repeated application of star–triangle transformations, and is quite similar in spirit to a transformation Baxter and Enting[15]) used, to demonstrate, in an elegant way, the relation between Ising models on hexagonal, triangular and square lattices. As a result of the similarity of these two transformations, the above relation is still implicitly present in our equations.

In section 3 we briefly discuss a symmetry property of the RG equations. In section 4, we analyse the fixed-point equations and obtain two fixed-point solutions. Due to the complexity of our equations, the linearized flow around an arbitrary fixed point is difficult to analyse. However, as argued in section 5, one would hope that there should exist one so-called local fixed point, the analogue of the one originally found by HSL, for which the linear flow problem simplifies. We actually find this fixed point by solving simple algebraic equations and obtain a unique eigenvalue $y_T = 1$, in agreement with the well-known exact result for square Ising models. In section 6 it is demonstrated that the renormalization transformation can be applied to square lattices with an oblique edge too. Finally, we discuss in section 7 some properties of differential real space renormalization theory connected with the concept of local fixed point, such as the relation between the dimensionality of the system and the number of parameters needed to set up the theory.

2. Construction of the RG equations

In this section we derive the RG equations for the Ising model on a square lattice. In subsection 2.1 we recapitulate some essential properties of the star–triangle transformation. In subsection 2.2 we describe the restructuring transformation that enables us to map the Hamiltonian $\mathscr{H}$ of a square Ising model onto the Hamiltonian $\mathscr{H}'$ of an Ising model with fewer degrees of freedom on an identical square lattice. The actual derivation of the renormalization equations is performed in subsection 2.3. The boundary conditions are discussed in subsection 2.4.

2.1. *The star–triangle transformation*

The well-known star–triangle transformation for Ising models, for a discussion of which we refer to ref. 16, can be summarized by the equation

$$\sum_{s'=\pm 1} \mathrm{e}^{s'(p_1 s_1 + p_2 s_2 + p_3 s_3)} = \mathrm{e}^{q_1 s_2 s_3 + q_2 s_3 s_1 + q_3 s_1 s_2 + g}, \tag{2.1}$$

which holds for any value of the Ising spins s_1, s_2 and s_3, provided that q_1, q_2, q_3 and g are given by

$$q_i = F(p_i, p_j, p_k), \quad i, j, k \text{ cyclic}, \tag{2.2}$$

where

$$F(p_1, p_2, p_3) := \frac{1}{4} \ln\left[\frac{\cosh(p_1 + p_2 + p_3)\cosh(-p_1 + p_2 + p_3)}{\cosh(p_1 - p_2 + p_3)\cosh(p_1 + p_2 - p_2)}\right], \tag{2.3}$$

and

$$g = \ln 2 + \tfrac{1}{4}\ln[\cosh(p_1 + p_2 + p_3)\cosh(-p_1 + p_2 + p_3)\cosh(p_1 - p_2 + p_3) \times \cosh(p_1 + p_2 - p_3)]. \tag{2.4}$$

In eq. (2.2) and throughout the rest of this paper, i, j, k cyclic stands for $i = 1$,

$j=2$, $k=3$ or any cyclic permutation. The star-triangle relation states that the contribution to the partition function from the summation over the states of the spin s' in the centre of the 'star' in fig. 1, can be accounted for by introducing interactions q_1, q_2 and q_3 of appropriate strength between the spins s_1, s_2 and s_3.

As in the analysis of HSL, we will frequently employ the symmetric matrix Q, the elements of which are defined by

$$Q_{il} := \frac{\partial F(p_i, p_j, p_k)}{\partial p_l}, \quad i, j, k \text{ cyclic}. \tag{2.5}$$

Explicit expressions for the matrix elements, some of which are given later in this paper, can be found in appendix A of HSL.

2.2. *A restructuring transformation consisting of star–triangle transformation cycles*

The basic idea of RG theory is to construct a mapping of the Hamiltonian $\mathcal{H}$ of a spin system, defined on a given lattice $\mathcal{L}$ onto a new Hamiltonian $\mathcal{H}'$ of a similar spin system on a lattice $\mathcal{L}'$, that contains fewer degrees of freedom. The mapping should be such that the free energies of the old and new systems are the same, and that the Hamiltonians $\mathcal{H}$ and $\mathcal{H}'$ are similar in form. The lattices $\mathcal{L}$ and $\mathcal{L}'$ we will consider in the first part of this paper are *semi-infinite* strips. In fig. 2a, the strip $\mathcal{L}$ has width L and lattice spacing a; the spins on this lattice are indicated by open circles. The new lattice $\mathcal{L}'$, the spins of which are indicated by crosses, has a somewhat larger lattice spacing $aL/(L-a)$, so that its boundaries coincide with those of $\mathcal{L}$. Obviously, it is possible to obtain the lattice $\mathcal{L}'$ by uniformly and isotropically stretching the lattice $\tilde{\mathcal{L}}$ of fig. 2b with respect to the spin that lies in the centre of the lowest row. According to the philosophy of differential real space renormalization, the interactions between the spins on the lattices are supposed to be slowly varying in space, i.e. to vary only appreciably over distances of the order L.

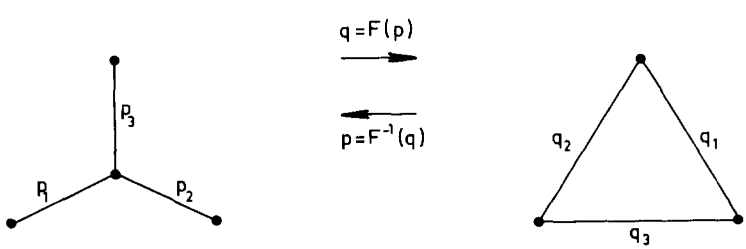

Fig. 1. The star of bonds p_i is converted by the star–triangle transformation into a triangle of interactions q_i. The relation between the p_i and q_i is expressed by eq. (2.2).

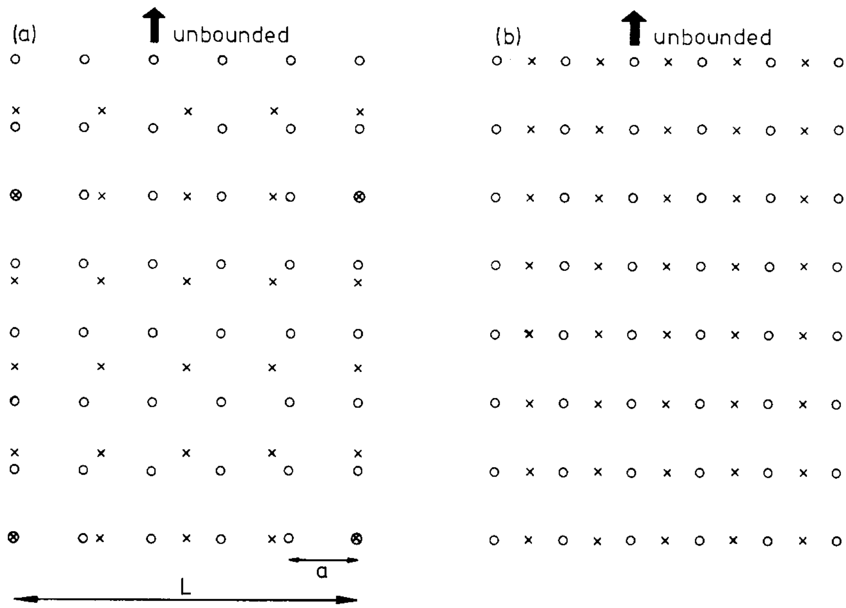

Fig. 2. (a) The lattices $\mathscr{L}$ (circles) and $\mathscr{L}'$ (crosses), which occupy the same spatial domain; (b) the lattices $\mathscr{L}$ (circles) and $\tilde{\mathscr{L}}$ (crosses). The lattice $\tilde{\mathscr{L}}$ goes over into $\mathscr{L}'$ by stretching it uniformly and isotropically with respect to the centre of the bottom row. All lattices are unbounded at the upper side.

As usual, it is convenient to construct the mapping from $\mathscr{H}$ onto $\mathscr{H}'$ in two steps:

I) a transformation of $\mathscr{H}$ onto the Hamiltonian $\tilde{\mathscr{H}}$ defined on $\tilde{\mathscr{L}}$;
II) a dilation of the coordinate system which yields $\mathscr{H}'$ from $\tilde{\mathscr{H}}$.

The use of semi-infinite strips rather than finite lattices as the starting point for the introduction of a renormalization transformation will entail some conceptual difficulties. We therefore emphasize that the only reason for doing so, is that the RG equations to be derived will turn out to fit this particular lattice shape better than other ones, which will be considered later.

We now proceed to show how sequences of star-triangle transformations can be used to perform step I explicitly. We successively transform one row of $\mathscr{L}$ into one of $\tilde{\mathscr{L}}$. The transformation of one row consists of essentially two steps and is sketched in fig. 3. We start in (a), where there are spins of both $\mathscr{L}$ and $\tilde{\mathscr{L}}$, connected to each other in one row by dashed bonds. First (step 1), by applying a star–triangle transformation to the spins of $\mathscr{L}$ in this row (circles), we obtain lattice (b), where the two square lattices are connected via dotted interactions which are drawn as the sides of downwards pointing triangles.

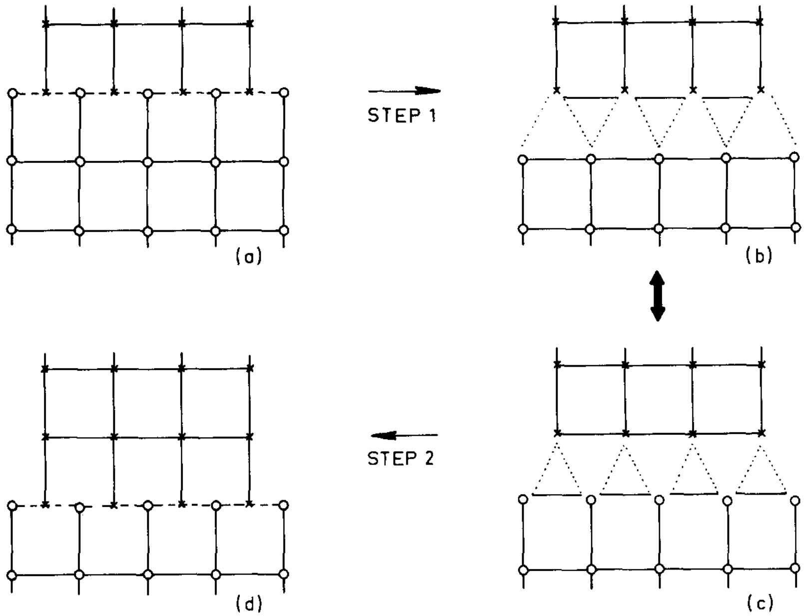

Fig. 3. Various stages in the transformation of one row: (a) initial situation; (b) a row of stars has been transformed into a row of triangles; (c) the interactions in the triangles have been regrouped; (d) by an inverse star–triangle transformation a new row of stars has been obtained.

Next, the bonds are regrouped by drawing the dotted bonds as the sides of upwards pointing triangles, as in (c). Finally, we perform an inverse star–triangle transformation to the triangles of (c) to obtain (d) (step 2). Comparison of figs. (a) and (d) shows that in each cycle we construct one row of $\tilde{\mathscr{L}}$ out of one of $\mathscr{L}$. Obviously, by applying these cycles of star–triangle transformations repeatedly, we can row by row build up a nearest neighbour Ising model on $\tilde{\mathscr{L}}$ out of that on $\mathscr{L}$.

One may wonder how the above transformation can start. To see this, imagine for the moment that we start with a large but finite strip of size $L \times L_v$ where L_v is the length in the vertical direction. We assume $L_v \gg L$. Then, at the upper row, we can start the transformation as indicated in fig. 4, by using a decoration transformation (a "truncated" inverse star–triangle transformation) for the horizontal interactions in the upper row. This yields fig. 4b, which is essentially the same as fig. 3a, and one may proceed as in the latter figure. The infinite strip is then obtained by taking the limit $L_v \to \infty$ while keeping L fixed. Of course, we can also start with the transformation depicted in fig. 4 at the bottom row and then move upwards through the semi-infinite strip. However, the latter procedure is beset with new problems which will be discussed later (section 7, remark 1).

Fig. 4. Initial stage of the transformation at the upper row of a finite lattice: (a) initial situation; (b) the horizontal interactions of the upper row have been transformed by a decoration transformation, $K_x = F(0, p_1, p_2)$.

We will denote the interactions on $\mathscr{L}$ (multiplied by $-1/kT$) by K_x and K_y, the interactions in the horizontal and vertical direction respectively. Similarly $\tilde{K}_x$ and $\tilde{K}_y$ are the interactions on $\tilde{\mathscr{L}}$. These quantities will be labelled by the position coordinate of the centre of the spins between which the respective interaction takes place. The dashed bonds will be denoted by p_1 and p_2 and the dotted ones by q_1 and q_2, and these will be labelled by the position coordinate of the cross-spin to which they belong; p_1 is the dashed bond which goes out from the cross-spin to the left, while q_1 is the dotted bond which goes out of a cross-spin to the right. With the aid of these conventions and the formulae given in subsection 2.1, step 1 of fig. 3 leads to the equations

$$\tilde{K}_x(\boldsymbol{R}) = F\left(K_y\left(\boldsymbol{R} - \frac{a}{2}\boldsymbol{e}_y\right), p_1\left(\boldsymbol{R} + \frac{a}{2}\boldsymbol{e}_x\right), p_2\left(\boldsymbol{R} - \frac{a}{2}\boldsymbol{e}_x\right)\right), \tag{2.6}$$

$$q_1(\boldsymbol{R}) = F\left(p_1(\boldsymbol{R} + a\boldsymbol{e}_x), p_2(\boldsymbol{R}), K_y\left(\boldsymbol{R} - \frac{a}{2}\boldsymbol{e}_y + \frac{a}{2}\boldsymbol{e}_x\right)\right), \tag{2.7}$$

$$q_2(\boldsymbol{R}) = F\left(p_2(\boldsymbol{R} - a\boldsymbol{e}_x), K_y\left(\boldsymbol{R} - \frac{a}{2}\boldsymbol{e}_x - \frac{a}{2}\boldsymbol{e}_y\right), p_1(\boldsymbol{R})\right). \tag{2.8}$$

Here, $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are unit vectors in the horizontal and vertical direction. Similarly, one finds that the following equations correspond with step 2 in fig. 3:

$$K_x(\boldsymbol{R}) = F\left(\tilde{K}_y\left(\boldsymbol{R} + \frac{a}{2}\boldsymbol{e}_y\right), p_1(\boldsymbol{R}), p_2(\boldsymbol{R})\right), \tag{2.9}$$

$$q_1(\boldsymbol{R}) = F\left(p_1(\boldsymbol{R} - a\boldsymbol{e}_y), p_2(\boldsymbol{R} - a\boldsymbol{e}_y), \tilde{K}_y\left(\boldsymbol{R} - \frac{a}{2}\boldsymbol{e}_y\right)\right), \tag{2.10}$$

$$q_2(\boldsymbol{R}) = F\left(p_2(\boldsymbol{R} - a\boldsymbol{e}_y), \tilde{K}_y\left(\boldsymbol{R} - \frac{a}{2}\boldsymbol{e}_y\right), p_1(\boldsymbol{R} - a\boldsymbol{e}_y)\right). \tag{2.11}$$

The equations for g (cf. eq. (2.1)) have not been given, since we will not discuss the differential equation for the free energy in this paper.

If we take the origin of the coordinate systems at the centre of the bottom row of spins, the renormalized interactions K'_x and K'_y on $\mathscr{L}'$ are obtained by means of a dilation of the coordinate system of $\tilde{\mathscr{L}}$ in the form

$$K'_\alpha([L/(L-a)]\boldsymbol{R}) = \tilde{K}_\alpha(\boldsymbol{R}), \quad \alpha = x, y. \tag{2.12}$$

Eqs. (2.6)–(2.12) are the fundamental equations of this renormalization transformation for square lattices; they will be analysed in the limit $a/L \to 0$ in the next section.

The above restructuring transformation was found by Hilhorst[14]). It is intimately related to a transformation introduced at about the same time by Baxter and Enting[15]). The common feature of these two transformations is that they both employ the star–triangle transformation repeatedly and proceed through the lattice. However, there are some striking differences too. Hilhorst's transformation modifies one row at a time and each row only once. Its most obvious application is in differential RG theory since it appears to be of hardly any use for homogeneous systems. The transformation of Baxter and Enting[15]), on the other hand, changes different rows a different number of times, and is therefore less easy to apply as a starting point to derive RG equations.

From the sequence of star–triangle transformations of Baxter and Enting[15]), it follows that the free energies of homogeneous hexagonal and triangular Ising lattices can be expressed in terms of the free energy of a homogeneous square Ising lattice,

$$\begin{array}{ccccc} \text{hexagonal,} & & \text{square,} & & \text{triangular} \\ \text{interactions} & \leftrightarrow & \text{interactions} & \leftrightarrow & \text{interactions} \\ p_1, p_2, p_3 & & p_3, F(p_3, p_1, p_2) & & q_i = F(p_i, p_j, p_k) \\ & & & & (i, j, k \text{ cyclic}) \end{array}. \tag{2.13}$$

This result, which will be reflected by our RG equations, will be of great help in guiding us in later stages of our calculations.

We finally remark that, although eqs. (2.6)–(2.12) relate interactions in the immediate neighbourhood of each other, we intuitively expect that, if the interactions K_x and K_y are changed in some region, p_1 and p_2 will be affected in that same region as well as (roughly) in a cone going out "downstream" from that region. Hence the transformation is expected to exhibit some non-local behaviour.

2.3. *Derivation of the differential equations*

We will now derive the flow equations for K_x and K_y in the limit $a/L \to 0$, assuming that all functions can be expanded in a Taylor series in a/L. As usual, we introduce the scaled coordinates $\boldsymbol{r} := x\boldsymbol{e}_x + y\boldsymbol{e}_y = \boldsymbol{R}/L$. From eqs.

(2.6) and (2.9) we obtain the "macroscopic equation"

$$K_x = F(K_y, p_1, p_2), \tag{2.14}$$

and, from the terms of order a/L,

$$\delta K_x + Q_{33}\delta K_y = \frac{a}{L}\left[-Q_{33}\frac{\partial K_y}{\partial y} + \frac{1}{2}Q_{31}\frac{\partial p_1}{\partial x} - \frac{1}{2}Q_{32}\frac{\partial p_2}{\partial x}\right]. \tag{2.15}$$

In this equation, $\delta K_x := \tilde{K}_x - K_x$ and $\delta K_y := \tilde{K}_y - K_y$, while the matrixelements of Q were defined in eq. (2.5). After elimination of q_1 and q_2 from eqs. (2.7), (2.8), (2.10) and (2.11), one obtains from the terms of order a/L in these equations

$$Q_{13}\delta K_y = \frac{a}{L}\left[Q_{11}\frac{\partial p_1}{\partial y} + Q_{12}\frac{\partial p_2}{\partial y} + Q_{11}\frac{\partial p_1}{\partial x} + \frac{1}{2}Q_{13}\frac{\partial K_y}{\partial x}\right], \tag{2.16}$$

$$Q_{23}\delta K_y = \frac{a}{L}\left[+Q_{21}\frac{\partial p_1}{\partial y} + Q_{22}\frac{\partial p_2}{\partial y} - Q_{22}\frac{\partial p_2}{\partial x} - \frac{1}{2}Q_{23}\frac{\partial K_y}{\partial x}\right]. \tag{2.17}$$

Elimination of δK_y from this equation yields the differential equation

$$\begin{aligned}(Q_{11}Q_{23} - Q_{21}Q_{13})\frac{\partial p_1}{\partial y} + (Q_{12}Q_{23} - Q_{22}Q_{13})\frac{\partial p_2}{\partial y}\\ + Q_{11}Q_{23}\frac{\partial p_1}{\partial x} + Q_{22}Q_{13}\frac{\partial p_2}{\partial x} + Q_{13}Q_{23}\frac{\partial K_y}{\partial x} = 0.\end{aligned} \tag{2.18}$$

To write this equation in a compact form, we define the vectors

$$\boldsymbol{b}_1 := -\frac{\boldsymbol{e}_x + \boldsymbol{e}_y}{2}, \quad \boldsymbol{b}_2 := \frac{\boldsymbol{e}_x - \boldsymbol{e}_y}{2}, \tag{2.19}$$

$$\boldsymbol{M}_i := \sum_{j=1}^{2} (-1)^j Q_{ji} Q_{(3-j)3}(\boldsymbol{b}_i + \boldsymbol{b}_j), \quad i = 1, 2, 3. \tag{2.20}$$

(The undefined vector $\boldsymbol{b}_3$, appearing in the definition of $\boldsymbol{M}_i$, drops out after performing the summation over j. Hence $\boldsymbol{b}_3$ may be any vector). It is also convenient to use the definitions

$$q_3 := K_x, \quad p_3 := K_y, \tag{2.21}$$

With these conventions, eq. (2.18) can be written as

$$\sum_{i=1}^{3} \boldsymbol{M}_i \cdot \nabla p_i = 0. \tag{2.22}$$

It is convenient to combine eqs. (2.15), (2.16) and (2.17). If one multiplies eq. (2.15) with Q_{33}^{-1} (the 33 – element of Q^{-1}, the inverse of Q), eq. (2.16) with Q_{31}^{-1} and eq. (2.17) with Q_{32}^{-1}, and adds the resulting equations, one obtains

$$\delta K_y + Q_{33}^{-1}\delta K_x = \frac{a}{L}\Big[\Big(Q_{31}^{-1}Q_{11} + \frac{1}{2}Q_{33}^{-1}Q_{31}\Big)\frac{\partial p_1}{\partial x} - \Big(Q_{32}^{-1}Q_{22} + \frac{1}{2}Q_{33}^{-1}Q_{32}\Big)\frac{\partial p_2}{\partial x}$$
$$+ \frac{1}{2}(Q_{31}^{-1}Q_{13} - Q_{32}^{-1}Q_{23})\frac{\partial K_y}{\partial x}$$
$$- Q_{33}^{-1}Q_{31}\frac{\partial p_1}{\partial y} - Q_{33}^{-1}Q_{32}\frac{\partial p_2}{\partial y} - Q_{33}^{-1}Q_{33}\frac{\partial K_y}{\partial y}\Big]. \tag{2.23}$$

Upon elimination of the derivatives of the p_i (p_1, p_2 and K_y) in favour of those of the q_i (q_1, q_2, K_x) by means of the relations

$$\nabla q_i = \sum_{j=1}^{3} Q_{ij}\nabla p_j, \quad \nabla p_i = \sum_{j=1}^{3} Q_{ij}^{-1}\nabla q_j, \tag{2.24}$$

eq. (2.23) reduces to

$$\delta K_y + Q_{33}^{-1}\delta K_x = \frac{a}{L}\Big[-Q_{33}^{-1}\frac{\partial K_y}{\partial y} + \frac{1}{2}Q_{31}^{-1}\frac{\partial q_1}{\partial x} - \frac{1}{2}Q_{32}^{-1}\frac{\partial q_2}{\partial x}\Big]. \tag{2.25}$$

Let us write $\delta t := a/L$; according to eqs. (2.12), (2.15) and (2.25), we then find for the total change $(K'_\alpha - K_\alpha)/\delta t$ in the limit $\delta t \to 0$

$$\frac{\partial K_x}{\partial t} + Q_{33}\frac{\partial K_y}{\partial t} = -Q_{33}\frac{\partial K_y}{\partial y} + \frac{1}{2}Q_{31}\frac{\partial p_1}{\partial x} - \frac{1}{2}Q_{32}\frac{\partial p_2}{\partial x} - \boldsymbol{r}\cdot\nabla K_x - Q_{33}\boldsymbol{r}\cdot\nabla K_y, \tag{2.26}$$

$$\frac{\partial K_y}{\partial t} + Q_{33}^{-1}\frac{\partial K_x}{\partial t} = -Q_{33}^{-1}\frac{\partial K_x}{\partial y} + \frac{1}{2}Q_{31}^{-1}\frac{\partial q_1}{\partial x} - \frac{1}{2}Q_{32}^{-1}\frac{\partial q_2}{\partial x} - \boldsymbol{r}\cdot\nabla K_y - Q_{33}^{-1}\boldsymbol{r}\cdot\nabla K_x. \tag{2.27}$$

These are the differential equations for the renormalization flow, written in a symmetric form. One may also show that eqs. (2.14) and (2.18) are equivalent to the equations obtained by replacing the p_i's by the q_i's, F by F^{-1} and Q by Q^{-1}. Nevertheless, we will from now on write the flow equations in a compact form that obscures this symmetry. We define

$$\boldsymbol{c}_1 := \frac{1}{2}\boldsymbol{e}_x, \quad \boldsymbol{c}_2 := -\frac{1}{2}\boldsymbol{e}_x, \quad \boldsymbol{c}_3 := -\boldsymbol{e}_y; \tag{2.28}$$

$$\boldsymbol{D}_{yi} := (1 - Q_{33}^{-1}Q_{33})^{-1}\sum_{j=1}^{3} Q_{3j}^{-1}Q_{ji}(\boldsymbol{c}_j - \delta_{3j}\boldsymbol{c}_i), \quad i = 1, 2, 3; \tag{2.29}$$

$$\boldsymbol{D}_{xi} := (1 - Q_{33}^{-1}Q_{33})^{-1}\Big[Q_{3i}\boldsymbol{c}_i - Q_{33}\sum_{j=1}^{3} Q_{3j}^{-1}Q_{ji}\boldsymbol{c}_j\Big], \quad i = 1, 2, 3. \tag{2.30}$$

Eqs. (2.14), (2.22), (2.26) and (2.27) are now equivalent to the following set of equations (henceforth, a summation over α or β runs over the values x and y, summations over i, j, etc. run from 1 to 3, unless specified otherwise)

$$K_x = F(K_y, p_1, p_2), \tag{2.31}$$

$$\sum_i \boldsymbol{M}_i \cdot \nabla p_i = 0, \tag{2.32}$$

$$\frac{\partial K_\alpha}{\partial t} = \sum_i \boldsymbol{D}_{\alpha i} \cdot \nabla p_i - \boldsymbol{r} \cdot \nabla K_\alpha, \quad \alpha = x, y. \tag{2.33}$$

Although the flow equation (2.33) is of the standard form found in differential renormalization, in that it relates $\partial K_\alpha/\partial t$ to the gradients of the interaction parameters, the present renormalization transformation is mathematically more intricate than that of HSL. For, in order to determine the flow for *given functions* K_x and K_y, we *first* have to solve eqs. (2.31) and (2.32) for p_1 and p_2; only after that $\partial K_\alpha/\partial t$ is known explicitly. Note also that since $p_1(\boldsymbol{r})$ and $p_2(\boldsymbol{r})$ have to be determined from a differential equation (supplemented with eq. (2.31)), they will not only depend on the values K_x and K_y at $\boldsymbol{r}$, but also on the whole shape of these functions. This entails the kind of non-locality already envisaged at the end of subsection 2.2.

As in the case of HSL, the number of degrees of freedom per unit area ρ decreases along a renormalization trajectory as $\rho(t) = \rho(0)\,\mathrm{e}^{-dt}$, where $d(=2)$ is the dimension. The eigenvalue y_T, to be determined, is therefore related to the critical exponent of the free energy $2-\alpha$ by $2-\alpha = d/y_T$.

2.4. *Boundary conditions*

Eqs. (2.31)–(2.33) have to be supplemented with boundary conditons in order to ensure that the transformation is also infinitesimal at the boundaries of the lattice. Since there is no freedom to choose p_1 and p_2 at will if we consider K_x and K_y to be given functions, the boundary conditions have to be formulated in terms of the latter interactions. Consider fig. 3a at the left boundary. In summing over the left boundary-spin of $\mathscr{L}$ (a circle) of the row to be transformed, the q_2 bond is obtained from a dedecoration transformation (a truncated star–triangle transformation): $q_2 = F(0, K_y, p_1)$. Nearby, in the bulk, we have (to order a/L) $q_2 \approx F(p_2, K_y, p_1)$. Hence no discontinuities arise if $F(0, K_y, p_1) \approx F(p_2, K_y, p_1)$. One may check that this equation (and the analogous one at the right boundary) is obeyed in each of the four following cases

$$\left.\begin{array}{l} \text{a) } K_x = 0 \\ \text{b) } K_x = \infty \\ \text{c) } K_y = 0 \\ \text{d) } K_y = \infty \end{array}\right\} \text{ at the left and right boundaries } (x = \pm\tfrac{1}{2}). \tag{2.34}$$

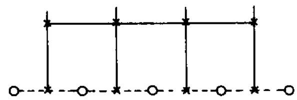

Fig. 5. The last stage of the transformation of the bottom row consists of summing over old spins (circles) which are only connected via bonds p_1 and p_2 to new ones.

The situation at the lower edge is somewhat different (see fig. 5). In this case, we have to formulate the boundary conditions such that one always has $F(K_y, p_1, p_2) \approx F(0, p_1, p_2)$. This equation is not fulfilled if $K_y \to \infty$, p_1, p_2 finite. On similar grounds, the case $K_x = 0$ has to be rejected, and we therefore only allow the following two cases,

$$\left.\begin{array}{l}\text{a) } K_x = \infty \\ \text{b) } K_y = 0\end{array}\right\} \text{at the lower edge } (y = 0). \tag{2.35}$$

Before, we discussed how we can start the transformation on a finite strip of size $L \times L_v$. In the first step, we then have a decoration transformation with $K_x = F(0, p_1, p_2)$. Clearly, in this case there is still a freedom to choose p_1 and p_2. We can e.g. prescribe the ratio p_1/p_2 at the upper row. This freedom may still be present after taking the limit $L_v \to \infty$, in that we obtain different transformations for different ratios $p_1(x, \infty)/p_2(x, \infty)$. The possible dependence of the transformation on this ratio will not be investigated further in this paper, however.

3. A symmetry property of the RG equation

In this section we closely follow HSL in using the symmetries of the RG equations to locate the critical subspace of these equations. We define $\bar{p}_j$ and $\bar{q}_j$, the variables dual to q_j and p_j respectively, by

$$\sinh 2\bar{p}_j := 1/\sinh 2q_j, \quad \sinh 2p_j =: 1/\sinh 2\bar{q}_j, \quad j = 1, 2, 3. \tag{3.1}$$

We will also use $\bar{K}_x := \bar{q}_3$, $\bar{K}_y := \bar{p}_3$. It is well known that the $\bar{q}_j$ are again the star–triangle transforms of the $\bar{p}_j$. HSL express the relations between these variables symbolically by writing

$$q = R_{\mathrm{ST}}(p), \quad p = R_{\mathrm{D}}(\bar{q}), \quad \bar{q} = R_{\mathrm{ST}}(\bar{p}), \quad \bar{p} = R_{\mathrm{D}}(q). \tag{3.2}$$

As discussed by HSL, points that are invariant under $R_{\mathrm{ST}}R_{\mathrm{D}}$ satisfy

$$q_i = \bar{q}_i, \quad i = 1, 2, 3, \tag{3.3}$$

$$p_i = \bar{p}_i, \quad i = 1, 2, 3, \tag{3.4}$$

$$v_1 + v_2 + v_3 = 1, \tag{3.5}$$

$$u_1 + u_2 + u_3 = 1, \tag{3.6}$$

where

$$v_i := (\sinh 2p_j \sinh 2p_k)^{-1}, \quad u_i := \sinh 2q_j \sinh 2q_k \quad (i, j, k \text{ cyclic}). \tag{3.7}$$

These equations are all equivalent. Note also that eqs. (3.1) and (3.4) imply that points which are invariant under $R_{\mathrm{ST}}R_{\mathrm{D}}$ also satisfy

$$\sinh 2K_x \sinh 2K_y = 1. \tag{3.8}$$

Upon transforming eqs. (2.31)–(2.33) along the lines set forth by HSL, one finds*

$$\bar{K}_x = F(\bar{K}_y, \bar{p}_1, \bar{p}_2), \tag{3.9}$$

$$\sum_i \boldsymbol{M}_i(\bar{p}) \cdot \nabla \bar{p}_i = 0, \tag{3.10}$$

$$\frac{\partial \bar{K}_\alpha}{\partial t} = \sum_i \boldsymbol{D}_{\alpha i}(\bar{p}) \cdot \nabla \bar{p}_i - \boldsymbol{r} \cdot \nabla \bar{K}_\alpha, \quad \alpha = x, y. \tag{3.11}$$

As in the case of HSL, the fact that the barred variables $\bar{K}_\alpha$ obey the same equations as the variables K_α implies that the renormalization operator commutes with the transformation $R_{\mathrm{ST}}R_{\mathrm{D}}$. Consequently, in the space of pairs of functions $(K_x(\boldsymbol{r}), K_y(\boldsymbol{r}))$, the subspace $\mathscr{C}^{\mathrm{s}}$,

$$\mathscr{C}^{\mathrm{s}} := \{(K_x(\boldsymbol{r}), K_y(\boldsymbol{r})) \mid \sinh 2K_x(\boldsymbol{r}) \sinh 2K_y(\boldsymbol{r}) = 1\}, \tag{3.12}$$

is not only invariant under $R_{\mathrm{ST}}R_{\mathrm{D}}$ but under the renormalization operator as well. It is the general picture of RG theory that there should exist a subspace of the parameter space that is invariant under the flow and of which the points are attracted under the flow by a fixed point. Accordingly, we associate the space $\mathscr{C}^{\mathrm{s}}$ with the critical subspace of the renormalization operator in the space of pairs of functions $(K_x(\boldsymbol{r}), K_y(\boldsymbol{r}))$. Indeed, since eq. (3.8) is the wellknown criticality condition for infinite homogeneous square Ising models, a pair of functions $(K_x, K_y) \in \mathscr{C}^{\mathrm{s}}$ describes an inhomogeneous system which is locally critical for all $\boldsymbol{r}$.

It will be convenient to introduce also the class $\mathscr{C}^{\mathrm{h}}$ of triplets of functions,

$$\mathscr{C}^{\mathrm{h}} := \{(p_1(\boldsymbol{r}), p_2(\boldsymbol{r}), p_3(\boldsymbol{r}) \mid \sum_i v_i = 1\}. \tag{3.13}$$

* It should be noted that eq. (3.9) is already contained in eq. (3.2) and that the symmetry of the flow equations of K_x and K_y can in fact most easily be established if they are written in the form (2.26), (2.27). Eq. (3.10) is derived in appendix A.

In view of the result of Baxter and Enting[15]), described in subsection 2.2, $\mathscr{C}^{\mathrm{h}}$ will be taken as the critical subspace of the "corresponding" hexagonal lattice. Note that the fact that $(K_x(\boldsymbol{r}), K_y(\boldsymbol{r})) \in \mathscr{C}^{\mathrm{s}}$ implies that the triplet $(p_1(\boldsymbol{r}), p_2(\boldsymbol{r}), p_3(\boldsymbol{r})) \in \mathscr{C}^{\mathrm{h}}$, and vice versa. Which particular element of $\mathscr{C}^{\mathrm{h}}$ corresponds to a given element of $\mathscr{C}^{\mathrm{s}}$ can, however, be determined only by solving eq. (2.32).

4. Derivation of two fixed-point solutions

In this section, we derive two fixed-point solutions. It should be mentioned that the analysis given below is not needed to arrive at the local fixed point (see section 5), but is included to illuminate some properties of the fixed-point equations and to show that there are fixed points other than the local one.

Up to now, we considered K_x and K_y to be given functions. In searching for fixed-point solutions, i.e. pairs $(K_x^*(\boldsymbol{r}), K_y^*(\boldsymbol{r}))$ and triplets $(p_1^*(\boldsymbol{r}), p_2^*(\boldsymbol{r}), p_3^*(\boldsymbol{r}))$ associated with them, for which

$$\sum_i \boldsymbol{D}^*_{\alpha i} \cdot \nabla p_i^* - \boldsymbol{r} \cdot \nabla K^*_\alpha = 0, \quad \alpha = x, y, \tag{4.1}$$

it is, however, more convenient to use the variables p_1^* and p_2^*. Let us define

$$S_1 := \sinh 2p_1, \quad S_2 := \sinh 2p_2, \quad S_3 := \sinh 2p_3. \tag{4.2}$$

Then, since fixed-point solutions lie in the critical subspace, $p_3(= K_y)$ can be obtained, once p_1 and p_2 are known, from the equation (cf. eq. (3.5))

$$S_3 = \frac{S_1 + S_2}{S_1 S_2 - 1}, \tag{4.3}$$

while K_x follows from the criticality-relation (3.8). Upon elimination of K_x and K_y from eq. (4.1), one finds (see appendix B) that the fixed-point equation becomes

$$\begin{aligned}(S_2^{*2} + 1)\left((x - \tfrac{1}{2})\frac{\partial S_1^*}{\partial x} + y\frac{\partial S_1^*}{\partial y}\right) + (S_1^{*2} + 1)\left((x + \tfrac{1}{2})\frac{\partial S_2^*}{\partial x} + y\frac{\partial S_2^*}{\partial y}\right)\\ + \frac{\partial S_1^*}{\partial x} + \frac{\partial S_1^*}{\partial y} - \frac{\partial S_2^*}{\partial x} + \frac{\partial S_2^*}{\partial y} = 0.\end{aligned} \tag{4.4}$$

In appendix B, we also show that eq. (2.32) reduces in the critical subspace $\mathscr{C}^{\mathrm{h}}$ to

$$\frac{\partial \ln S_1 S_2}{\partial x} - \frac{\partial \ln S_2/S_1}{\partial y} = 0. \tag{4.5}$$

Of course, fixed-point solutions of eq. (4.4) should also obey this equation. To arrive at the proper boundary conditions for S_1 and S_2, we note that for pairs of functions $(K_x(r), K_y(r)) \in \mathscr{C}^s$, boundary conditions (2.34a) and (2.34d) are equivalent, and so are (2.34b) and (2.34c). In the first case, we assume†

$$S_1 = \infty, \quad S_2 = 0, \quad \text{for } x = -\tfrac{1}{2}; \quad S_1 = 0, \quad S_2 = \infty, \quad \text{for } x = \tfrac{1}{2}, \tag{4.6}$$

whereas in the second case, we need to have

$$S_1 = \infty, \quad S_2 = \infty, \quad \text{for } x = \pm\tfrac{1}{2}. \tag{4.7}$$

At the lower edge, (2.35a) and (2.35b) are again equivalent. The boundary condition therefore is

$$S_1 = \infty, \quad S_2 = \infty, \quad \text{for } y = 0. \tag{4.8}$$

We first analyse eq. (4.5). If we consider $\ln S_1/S_2$ and $\ln S_1S_2$ as the components of a vector $(\ln S_2/S_1, \ln S_1S_2)$, eq. (4.5) shows that the curl of this vector should be zero. Consequently, S_1 and S_2 can be expressed in terms of a single "*potential*" V by writing $(\ln S_2/S_1, \ln S_1S_2) = 2(\partial V/\partial x, \partial V/\partial y)$, so that all functions S_1 and S_2 in the critical subspace are of the form

$$S_1 = \exp\left(-\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right), \quad S_2 = \exp\left(\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right). \tag{4.9}$$

As I have not been able to find the general solution of the partial differential equation for V^* one gets by substitution of the expressions (4.9) into eq. (4.4), we will, instead of analyzing this general equation, derive fixed-point solutions starting from the ansatz

$$S_1^* = m(y)M_1(x-y)n(x), \quad S_2^* = m(y)M_2(x+y)/n(x). \tag{4.10}$$

These function are obvious solutions of eq. (4.5). The functions M_1 and M_2 will eventually be set equal to 1 for the semi-infinite strip. They are retained for the time being in the discussion, however, with a view to a later analysis of lattices with an oblique edge (section 6). Note that if $M_1 = M_2 = 1$, the ansatz (4.10) appears to be the simplest one that is compatible with boundary conditions (4.6) and (4.8), and that eq. (4.7) can not be obeyed by functions of this form. It is also clear from eq. (4.8) that $m(y) \to \infty$ for $y \downarrow 0$. Let us therefore assume that $m(y)$ behaves as $m(y) = m_{ns}(y)/y^\gamma$ where $m_{ns}(y)$ is a nonsingular function with finite limit for $y = 0$, and where γ is positive. If we substitute these expressions into eq. (4.4), we can order the various terms in

† There are more possibilities, all consistent with eqs. (2.34a), (2.34d) and eq. (4.3). Anticipating later results, we have left the case $S_1 = 0$, $S_2 = \infty$ for $x = -\frac{1}{2}$ and similar conditions for $x = \frac{1}{2}$ out of consideration, as well as the case S_1, S_2 finite, $S_1S_2 = 1$.

powers of $y^{-\gamma}$. Clearly, it is necessary that the pre-factor of the most divergent terms vanish in the limit $y \downarrow 0$. For $0<\gamma<\frac{1}{2}$, this turns out to be impossible, whereas for $\gamma \geq \frac{1}{2}$, this is indeed the case provided that

$$\text{I)}\ \gamma > \frac{1}{2}:\quad \gamma(N_1(x)+N_2(x)) + \left(x-\frac{1}{2}\right)\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} + \left(x+\frac{1}{2}\right)\frac{\mathrm{d}N_2}{\mathrm{d}x} = 0, \tag{4.11}$$

$$\text{II)}\ \gamma = \frac{1}{2}:\quad \frac{1}{2}(N_1(x)+N_2(x)) + \frac{1}{2}m_{ns}^{-2}(0)$$
$$\times (N_1^2(x)N_2(x) + N_2^2(x)N_1(x)) + \left(x-\frac{1}{2}\right)\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} + \left(x+\frac{1}{2}\right)\frac{\mathrm{d}N_2}{\mathrm{d}x} = 0, \tag{4.12}$$

where we have defined

$$N_1(x) := \lim_{y\downarrow 0}(M_1(x-y)n(x))^{-1}, \quad N_2(x) := \lim_{y\downarrow 0}(M_2(x+y)/n(x))^{-1}. \tag{4.13}$$

I) We first investigate eq. (4.11). Since it allows classes of solutions, we define

$$R(x) := N_2(x)/N_1(x), \tag{4.14}$$

and consider $R(x)$ to be a given function. Upon elimination of N_2 from eq. (4.12), one gets

$$N_1(x)\left(\gamma(1+R(x)) + \left(x+\frac{1}{2}\right)\frac{\mathrm{d}R(x)}{\mathrm{d}x}\right) + \left(x-\frac{1}{2}+\left(x+\frac{1}{2}\right)R(x)\right)\frac{\mathrm{d}N_1(x)}{\mathrm{d}x} = 0. \tag{4.15}$$

This equation is most simple if $\gamma = 1$, and we will therefore restrict the analysis to this case. For $\gamma = 1$, eq. (4.15) admits two different types of solutions:

a) If $R(x) = (\frac{1}{2}-x)/(\frac{1}{2}+x)$, eq. (4.15) with $\gamma = 1$ is obeyed whatever $N_1(x)$ is. Hence, every set of functions $N_1(x)$ and $N_2(x)$ for which

$$N_2(x)/N_1(x) = (\tfrac{1}{2}-x)/(\tfrac{1}{2}+x), \tag{4.16}$$

constitutes a solution of eq. (4.11).

b) If $R(x) \neq (\frac{1}{2}-x)/(\frac{1}{2}+x)$, the solution of eq. (4.15) with $\gamma = 1$ is

$$N_1(x) = c/|x-\tfrac{1}{2}+(x+\tfrac{1}{2})R(x)|, \tag{4.17}$$

where c is an arbitrary constant.

Let us now put $M_1 = M_2 = 1$, so that $N_2(x) = N_1^{-1}(x) = R^{1/2}(x) = n(x)$. It turns out that in this case solutions of the type (4.17) are not compatible with the boundary conditions (4.6); however, eq. (4.16) now yields

$$n(x) = \left(\frac{\frac{1}{2}-x}{\frac{1}{2}+x}\right)^{1/2}, \tag{4.18}$$

and with this function the boundary condition (4.6) is obeyed.

II) Next, we investigate eq. (4.12). If we again put $M_1 = M_2 = 1$, so that $N_1(x)N_2(x) = 1$, we find that eq. (4.12) reduces to eq. (4.11), provided that $m_{ns}(0) = 1$. Hence, eq. (4.18) is also a solution of eq. (4.12) in case $m_{ns}(0) = 1$.

If S_1^* and S_2^* are of the type (4.10) with $M_1 = M_2 = 1$, then $S_1^* S_2^* = m^2$. Since S_3^* must be positive, eq. (4.3) shows that it is therefore necessary that $m > 1$. If we substitute the expressions (4.10) with $n(x)$ given by eq. (4.18) and $M_1 = M_2 = 1$ into eq. (4.4), this equation reduces to

$$\frac{1}{(\frac{1}{2}-x)^{1/2}(\frac{1}{2}+x)^{1/2}}\left[m(y)(m^2(y)-1)+\{y(1+m^2(y))+1\}\frac{\mathrm{d}m(y)}{\mathrm{d}y}\right]=0. \qquad (4.19)$$

Hence, fixed-point solutions can now be found by solving an ordinary differential equation for $m(y)$ with the boundary-condition $m(0) = \infty$. Using the fact that $m(y) > 1$, it follows from eq. (4.19) that $m(y)$ decreases monotonically and that $m(y)$ approaches 1 asymptotically as $m(y) \simeq 1 + \text{const}/y$ for $y \to \infty$. If we substitute $m(y) = m_{ns}(y)/y^\gamma$ in this equation, it turns out that only the cases $\gamma = \frac{1}{2}$, $m_{ns}(0) = 1$ and $\gamma = 1$ are allowed, as expected. We found two simple solutions in agreement with these asymptotic results, viz.

$$m(y) = 1 + 1/y \qquad (4.20)$$

and

$$m(y) = (1 + 1/y)^{1/2}. \qquad (4.21)$$

The corresponding fixed-point solutions are

$$S_1^* = \frac{(y+1)(\frac{1}{2}-x)^{1/2}}{y(\frac{1}{2}+x)^{1/2}}, \quad S_2^* = \frac{(y+1)(\frac{1}{2}+x)^{1/2}}{y(\frac{1}{2}-x)^{1/2}}, \quad S_3^* = \frac{y(y+1)}{(2y+1)(\frac{1}{2}-x)^{1/2}(\frac{1}{2}+x)^{1/2}}, \qquad (4.22)$$

and

$$S_1^* = \left[\frac{(y+1)(\frac{1}{2}-x)}{y(\frac{1}{2}+x)}\right]^{1/2}, \quad S_2^* = \left[\frac{(y+1)(\frac{1}{2}+x)}{y(\frac{1}{2}-x)}\right]^{1/2}, \quad S_3^* = \left[\frac{y(y+1)}{(\frac{1}{2}-x)(\frac{1}{2}+x)}\right]^{1/2}. \qquad (4.23)$$

In the next section, it will turn out that the solution (4.23) constitutes the so-called local fixed-point solution.

5. The linearized flow operator and the critical exponent

In this section we study the linearized flow around fixed points in the unstable (temperature-like) direction. The analysis of the linearized flow is complicated by the problem that we have no explicit expression for the linearized flow operator at our disposal as long as we have not solved the linearized equations corresponding to eqs. (2.31) and (2.32). In subsection 5.1 we state this problem in more detail and argue that there should be one particular fixed point, the "local" fixed-point of the renormalization transformation, for which we may circumvent solving eq. (2.32) explicitly to obtain the linearized flow. In subsection 5.2 we collect all the necessary formulae on the basis of which we obtain the local fixed point and the critical exponent y_T in subsection 5.3.

5.1. *The general approach*

For the discussion of the evolution of small deviations $\psi_\alpha(\boldsymbol{r}) := \delta K_\alpha(\boldsymbol{r})$ form the fixed-point solution $K_\alpha^*(\boldsymbol{r})$, it will be convenient to write the linearized flow-equations in the form

$$\frac{\partial\psi_\alpha}{\partial t} = \sum_\beta T_{\alpha\beta}(\boldsymbol{r}, \boldsymbol{\nabla})\psi_\beta, \quad \alpha = x, y, \tag{5.1}$$

where $T(\boldsymbol{r}, \boldsymbol{\nabla})$ is a linear operator. The linearized version of eq. (2.33) reads

$$\frac{\partial\psi_\alpha}{\partial t} = \sum_i \boldsymbol{D}_{\alpha i}^* \cdot \boldsymbol{\nabla}\Psi_i + \sum_{ij} \left(\frac{\partial \boldsymbol{D}_{\alpha i}}{\partial p_j}\right)^* \cdot \boldsymbol{\nabla} p_i^* \Psi_j - \boldsymbol{r} \cdot \boldsymbol{\nabla}\psi_\alpha, \quad \alpha = x, y. \tag{5.2}$$

From here on, $\Psi_i(\boldsymbol{r}) := \delta p_i(\boldsymbol{r})$ $(i = 1, 2, 3)$ denotes the deviation from the fixed-point solution p_i^* corresponding to the deviation $\psi_\alpha(\boldsymbol{r})$ from $K_\alpha^*(\boldsymbol{r})$. By identifying the right-hand sides of eqs. (5.1) and (5.2), we only get a formal expression for T, since Ψ_i is still unknown. To obtain an explicit expression for T, we have to determine δp_1 and δp_2 for arbitrary δK_α from the eqs. (2.31) and (2.32), linearized around the fixed-point solution. This set of equations is of the form

$$\sum_{j=1}^{2} I_{ij}(\boldsymbol{r}, \boldsymbol{\nabla})\delta p_j(\boldsymbol{r}) = \sum_\alpha H_{i\alpha}(\boldsymbol{r}, \boldsymbol{\nabla})\delta K_\alpha(\boldsymbol{r}), \quad i = 1, 2. \tag{5.3}$$

Here I and H are known linear operators. For arbitrary δK_α, the solution of this equation is

$$\delta p_i(\boldsymbol{r}) = \sum_\alpha \sum_{j=1}^{2} \int \mathrm{d}\boldsymbol{r}' G_{ij}(\boldsymbol{r} \mid \boldsymbol{r}') H_{j\alpha}(\boldsymbol{r}', \boldsymbol{\nabla}')\, \delta K_\alpha(\boldsymbol{r}'), \quad i = 1, 2, \tag{5.4}$$

where the Green's functions G_{ij} are the solutions of

$$\sum_{j=1}^{2} I_{ij}(\boldsymbol{r}, \boldsymbol{\nabla}) G_{jk}(\boldsymbol{r} \mid \boldsymbol{r}') = \delta_{ik}\delta(\boldsymbol{r} - \boldsymbol{r}'), \quad i = 1, 2. \tag{5.5}$$

A direct attack along these lines looks rather cumbersome. To get round the above program, we make the following observations:

1) As discussed $K := (K_x(\boldsymbol{r}), K_y(\boldsymbol{r})) \in \mathscr{C}^s$ implies that $p := (p_1(\boldsymbol{r}), p_2(\boldsymbol{r}), p_3(\boldsymbol{r})) \in \mathscr{C}^h$. Consequently, if $\psi(\boldsymbol{r})$ lies in the linear subspace tangent to $\mathscr{C}^s$ at K^* then $\Psi(\boldsymbol{r})$ lies in the linear subspace tangent to $\mathscr{C}^h$ at p^*. There should then be a linear relation between the projections $\psi^p(\boldsymbol{r})$ of $\psi(\boldsymbol{r})$ and $\Psi^p(\boldsymbol{r})$ of $\Psi(\boldsymbol{r})$ onto the subspaces orthogonal to the above tangent spaces, that enables us to calculate $\Psi^p(\boldsymbol{r})$ for given $\psi^p(\boldsymbol{r})$ without analysing eq. (2.32).

2) HSL showed that it is convenient for the study of unstable (temperaturelike) perturbations, to consider the eigenvalue problem for the (properly defined) adjoint operator $\tilde{T}(\boldsymbol{r}, \boldsymbol{\nabla})$ of the linearized flow operator $T(\boldsymbol{r},\boldsymbol{\nabla})$. Though the $\tilde{T}(\boldsymbol{r}, \boldsymbol{\nabla})$ of HSL was at first sight a differential operator, it turned out to reduce to the unit operator when acting on temperaturelike perturbations around their particular fixed point (which implied that the spectrum of eigenfunctions was infinitely degenerate with eigenvalue $y_T = 1$).

However, one may turn the argument around. Without already knowing a fixed-point solution, one may *require* that the gradient terms in the expression for $\tilde{T}$ (which now depends on the unknown function p_1^*, p_2^*, etc.) drop out, so that $\tilde{T}$ is not a proper differential operator any more. This requirement will lead to simple algebraic equations for the functions p_1^*, p_2^*, etc. One then just has to *check* whether the set of functions solving these equations does *indeed* constitute a fixed-point solution of the flow equations. If it does, we will call this fixed point the *local fixed point* of the renormalization transformation. This name stems from the fact that, as we will see, we can study the linearized flow around this fixed point in the temperature direction locally so that we may identify the resulting critical properties with those of homogeneous systems. Following this line of argument, one can indeed check for the equations of HSL that their original fixed point is, in fact, the only local fixed point†.

3) If the linearized flow in the temperature direction around the local fixed point is to be associated with the critical properties of homogeneous systems, then the knowledge of Ψ^p of Ψ alone (cf. observation 1) should suffice to calculate the eigenvalue y_T at the local fixed point. To see this, consider a small perturbation ψ_A at position $\boldsymbol{r}_A$ in the temperature direction. In general, the corresponding perturbation Ψ_A will have a non-zero component, Ψ_A^t say, in the linear subspace tangent to $\mathscr{C}^h$ at p^*. We compare this situation with the one in which there is in addition another perturbation at $\boldsymbol{r}_B$ that lies in the subspace tangent to $\mathscr{C}^s$ at K^*. By virtue of the differential equation, obtained

†Thus, the fixed-points found by Knops and Hilhorst[3]) are non-local ones, and it may be of interest to study temperaturelike perturbations around these fixed points; cf. remark 4 in section 7.

by linearizing eq. (2.32) around the local fixed-point solution, the perturbation at r_B changes Ψ'_A by an amount Ψ'_{AB} in comparison with the first situation. The connection of the flow in the temperature direction at r_A with the critical properties of a homogeneous system is only possible if this flow does not depend on Ψ'_A. For, if it did depend on Ψ'_A, it would also depend on Ψ'_{AB}, and consequently the eigenvalue y_T at r_A would be modified by the presence of the perturbation at r_B; obviously, such a dependence would prevent us from identifying y_T at r_A with the critical properties of a homogeneous Ising system that has at criticality interaction parameters $K_x = K_x^*(r_A)$ and $K_y = K_y^*(r_A)$.

In conclusion, the above observations suggest to search for the local fixed-point solution in order to circumvent solving eq. (2.32), linearized around the fixed-point solution. This local fixed point should be signaled by the fact that $\tilde{T}$ neither is a proper differential operator, nor is determined by Ψ'. We actually carry out this program in subsection 5.3. Some results needed in that analysis are derived in the next subsection.

5.2. *Some properties of* $\boldsymbol{D}$

In this subsection we derive a number of results that are related to the fact that $\mathscr{C}^s$ is invariant under the flow. Of main importance are eq. (5.25) and two properties of $\boldsymbol{D}$, expressed by eqs. (5.23) and (5.30). We first list the expressions for the elements of Q and its inverse, valid in the critical subspace. According to eqs. (A.9), (A.11) and (A.12) of HSL, these are

$$Q_{ii} =: Q_0 = -1/S_1S_2S_3, \quad \forall p \in \mathscr{C}^h, \quad i = 1,2,3, \tag{5.6}$$

$$Q_{jk} = -Q_0C_i, \quad \forall p \in \mathscr{C}^h, \quad i,j,k \text{ cyclic}, \tag{5.7}$$

$$Q_{ii}^{-1} = Q_0S_i^2, \quad \forall p \in \mathscr{C}^h, \quad i = 1,2,3, \tag{5.8}$$

$$Q_{jk}^{-1} = -Q_0C_iS_jS_k, \quad \forall p \in \mathscr{C}^h, \quad i,j,k \text{ cyclic}. \tag{5.9}$$

The variables S_i were defined in eq. (4.2), and the C_i are defined

$$C_i := \cosh 2p_i, \quad i = 1,2,3. \tag{5.10}$$

Two useful relations that follow from eqs. (3.5) and (3.7) are (cf. eq. (A.7) of HSL)

$$S_i = (S_j + S_k)/(S_jS_k - 1), \quad \forall p \in \mathscr{C}^h, \quad i,j,k, \text{cyclic}, \tag{5.11}$$

$$C_i = C_jC_k/(S_jS_k - 1), \quad \forall p \in \mathscr{C}^h, \quad i,j,k \text{ cyclic}. \tag{5.12}$$

The normal vector $\zeta = (\zeta_x, \zeta_y)$ of the critical surface with equation

$\sinh 2K_x \sinh 2K_y = 1$ is defined by

$$\zeta_\alpha := A_1 \frac{\partial}{\partial K_\alpha} \sinh 2K_x \sinh 2K_y \,\Big|_{\sinh 2K_x \sinh 2K_y = 1}, \quad \alpha = x, y. \tag{5.13}$$

Here A_1 is a normalization constant. As it is convenient to write all quantities in terms of p_1, p_2 and $p_3(= K_y)$, we write ζ in the form

$$\zeta = (\zeta_x, \zeta_y) = \frac{1}{C_3}(S_3, 1). \tag{5.14}$$

Similarly, the normal vector $\eta = (\eta_1, \eta_2, \eta_3)$ of the surface with equation $\Sigma_i v_i = 1$ is defined by (A_2 is another normalization constant)

$$\eta_i := A_2 \frac{\partial}{\partial p_i} \sum_j v_j \,\Big|_{\sum_j v_j = 1}, \qquad i = 1, 2, 3. \tag{5.15}$$

With the aid of eq. (5.12), one finds for η

$$\eta_i = -A^{-1} C_j C_k, \quad i, j, k \text{ cyclic}, \tag{5.16}$$

where

$$A := (C_1^2 C_2^2 + C_2^2 C_3^2 + C_3^2 C_1^2)^{1/2}. \tag{5.17}$$

From eqs. (5.6), (5.7), (5.14) and (5.16) it is found that ζ and η are related by*

$$\zeta_x Q_{3i} + \zeta_y \delta_{3i} = \mu_0 \eta_i, \quad \forall p \in \mathscr{C}^{\mathrm{h}}, \quad i = 1, 2, 3, \tag{5.18}$$

where

$$\mu_0 := -A/S_1 S_2 C_3^2. \tag{5.19}$$

For all $K \in \mathscr{C}^{\mathrm{s}}$ and all $p \in \mathscr{C}^{\mathrm{h}}$, we have

$$\sum_\alpha \zeta_\alpha \nabla K_\alpha = 0, \quad \forall K \in \mathscr{C}^{\mathrm{s}}, \tag{5.20}$$

$$\sum_i \eta_i \nabla p_i = 0, \quad \forall p \in \mathscr{C}^{\mathrm{h}}. \tag{5.21}$$

The fact that $\mathscr{C}^{\mathrm{s}}$ is invariant under the flow (cf. section 3) implies that $\Sigma_\alpha \zeta_\alpha \partial K_\alpha / \partial t = 0$ for all $K \in \mathscr{C}^{\mathrm{s}}$. From eqs. (2.33) and (5.20), we see that this is only possible if

$$\sum_\alpha \sum_i \zeta_\alpha \boldsymbol{D}_{\alpha i} \cdot \nabla p_i = 0, \quad \forall p \in \mathscr{C}^{\mathrm{h}}. \tag{5.22}$$

* The corresponding relation between η and the normal of the critical surface of the triangular lattice is given in eq. (4.29) of HSL.

We emphasise that this result should hold for *all* triplets of functions $p \in \mathscr{C}^{\mathrm{h}}$, irrespective of whether they satisfy eq. (2.32) or not. For, eqs. (2.33) and (2.31) form a consistent set of equations that leave $\mathscr{C}^{\mathrm{s}}$ invariant if supplemented with *any* extra equation for the p_i which is unchanged by transforming the p_i to the $\bar{p}_i$. Eq. (5.22) can only be obeyed for all $p \in \mathscr{C}^{\mathrm{h}}$ if there is a vector $\boldsymbol{\mu}$ such that

$$\sum_\alpha \zeta_\alpha \boldsymbol{D}_{\alpha i} = \mu_0 \boldsymbol{\mu} \eta_i, \quad \forall p \in \mathscr{C}^{\mathrm{h}}, \quad i = 1, 2, 3. \tag{5.23}$$

The factor μ_0 in the right-hand side of this equation is introduced for convenience. On the basis of the explicit expressions for $\zeta, \eta, \boldsymbol{D}, Q$ and μ_0, one finds for $\boldsymbol{\mu}$

$$\boldsymbol{\mu} = \frac{1}{2}\frac{S_2 - S_1}{S_1 + S_2}\boldsymbol{e}_x + \frac{S_3}{S_1 + S_2}\boldsymbol{e}_y. \tag{5.24}$$

The property of $\boldsymbol{D}$ expressed by eq. (5.23) is the analogue of the result (4.22) of HSL.

We decompose small deviations ψ from a fixed-point solution by writing $\psi = \psi^p + \psi^t$, where ψ^p is parallel to ζ^* and ψ^t perpendicular to it. Similarly we decompose the corresponding Ψ into Ψ^p and Ψ^t, the components parallel and perpendicular to η^*, respectively. The linear relation between ψ^p and Ψ^p, alluded to in the previous subsection, reads (cf. eq. (5.18))

$$\Psi_i^p = \mu_0^{*-1}\Big(\sum_\alpha \zeta_\alpha^* \psi_\alpha\Big)\eta_i^*, \quad i = 1, 2, 3. \tag{5.25}$$

For later reference, we derive a relation involving Ψ^t; in the linear subspace tangent to $\mathscr{C}^{\mathrm{h}}$ at p^*, one has according to eq. (5.23)

$$\sum_\alpha \delta\zeta_\alpha \boldsymbol{D}_{\alpha i}^* + \sum_\alpha \sum_j \zeta_\alpha^* \left(\frac{\partial \boldsymbol{D}_{\alpha i}}{\partial p_j}\right)^* \Psi_j^t = \delta(\mu_0 \boldsymbol{\mu})\eta_i^* + \mu_0^* \boldsymbol{\mu}^* \delta\eta_i, \quad i = 1, 2, 3. \tag{5.26}$$

Here $\delta\zeta_\alpha$ denotes the change in ζ_α due to the variation Ψ^t, etc. If we contract this equation with ∇p_i^* and use eq. (4.1) as well as eq. (5.21), specialised to the fixed point solution, we get

$$\sum_\alpha \sum_{ij} \zeta_\alpha^* \left(\frac{\partial \boldsymbol{D}_{\alpha i}}{\partial p_j}\right)^* \cdot \nabla p_i^* \Psi_j^t = \sum_i \mu_0^* \boldsymbol{\mu}^* \cdot \nabla p_i^* \delta\eta_i - \sum_\alpha \boldsymbol{r} \cdot \nabla K_\alpha^* \delta\zeta_\alpha. \tag{5.27}$$

Similarly, eq. (5.18) yields for these variations

$$\delta\zeta_x Q_{3i}^* + \delta\zeta_y \delta_{3i} + \zeta_x^* \delta Q_{3i} = \delta\mu_0 \eta_i^* + \mu_0^* \delta\eta_i, \quad i = 1, 2, 3. \tag{5.28}$$

By eliminating $\delta\eta_i$ from eq. (5.27) with the aid of this result and using also the fact that

$$\sum_i \delta Q_{3i} \nabla p_i^* = \sum_{ij} \left(\frac{\partial^2 F(p_3, p_1, p_2)}{\partial p_j \partial p_i} \right)^* \Psi_j^t \nabla p_i^* = \sum_j \Psi_j^t \nabla Q_{3j}^*, \tag{5.29}$$

one finally obtains

$$\sum_\alpha \sum_{ij} \zeta_\alpha^* \left(\frac{\partial D_{\alpha i}}{\partial p_j} \right)^* \cdot \nabla p_i^* \Psi_j^t = \sum_\alpha (\mu^* - r) \cdot \nabla K_\alpha^* \delta\zeta_\alpha + \sum_i \zeta_x^* \mu^* \cdot \nabla Q_{3i}^* \Psi_i^t. \tag{5.30}$$

5.3. *The local fixed point and the critical exponent*

For two arbitrary deviations ϕ and ψ from the fixed-point solution K^* we define an inner product

$$\langle \phi, \psi \rangle = \int \mathrm{d}r \sum_\alpha \phi_\alpha(r) \psi_\alpha(r). \tag{5.31}$$

Here, the integral is taken over the domain of the lattice; ϕ and ψ are assumed to vanish sufficiently rapid at large y to ensure the convergence of the integral. The reason to study the adjoint operator $\tilde{T}$ is that the subspace consisting of functions ϕ, such that

$$\phi_\alpha(r) = f(r)\zeta_\alpha^*(r), \quad \alpha = x, y, \tag{5.32}$$

is invariant under $\tilde{T}$. For, in view of the fact that $\mathscr{C}^s$ is invariant under the flow, the linear subspace tangent to $\mathscr{C}^s$ at K^*, which consists of all functions ψ for which

$$\sum_\alpha \zeta_\alpha^*(r) \psi_\alpha(r) = 0, \quad \text{for all } r, \tag{5.33}$$

is invariant under T; since this subspace is also orthogonal to the subspace defined by eq. (5.32), the latter is invariant under $\tilde{T}$. For the study of temperaturelike perturbations of the type (5.32), we therefore wish to consider the eigenvalue problem

$$\sum_\alpha \tilde{T}_{\beta\alpha} f \zeta_\alpha^* = y_T f \zeta_\beta^*, \quad \beta = x, y. \tag{5.34}$$

The action of $\tilde{T}$ on $f\zeta^*$ is defined through the equation $\langle \tilde{T} f\zeta^*, \psi \rangle = \langle f\zeta^*, T\psi \rangle$ for all ψ. From eq. (5.2) we obtain for arbitrary ψ†

† We assume that the boundary-term arising from the integration by parts vanishes. In general, this leads to certain restrictions on ψ and f; at the local fixed-point however, these boundary terms vanish identically for all ψ and f.

$$\langle \tilde{T} f \zeta^*, \psi \rangle = \sum_\alpha \int \mathrm{d}r f \zeta_\alpha^* \Big[\sum_i \boldsymbol{D}_{\alpha i}^* \cdot \nabla \Psi_i + \sum_{ij} \Big(\frac{\partial \boldsymbol{D}_{\alpha i}}{\partial p_j} \Big)^* \cdot \nabla p_i^* \Psi_j - \boldsymbol{r} \cdot \nabla \psi_\alpha \Big]$$

$$= \sum_\alpha \int \mathrm{d}\boldsymbol{r} \Big[- \sum_i \nabla \cdot (f \zeta_\alpha^* \boldsymbol{D}_{\alpha i}^*) \Psi_i + \sum_{ij} f \zeta_\alpha^* \Big(\frac{\partial \boldsymbol{D}_{\alpha i}}{\partial p_j} \Big)^* \cdot \nabla p_i^* (\Psi_j^{\mathrm{p}} + \Psi_j^{\mathrm{t}}) + \nabla \cdot (f \zeta_\alpha^* \boldsymbol{r}) \psi_\alpha \Big]. \quad (5.35)$$

The first term between square brackets can be rewritten with the aid of eqs. (5.23) and (5.18),

$$\sum_\alpha \sum_i \nabla \cdot (f \zeta_\alpha^* \boldsymbol{D}_{\alpha i}^*) \Psi_i = \sum_i \nabla \cdot [f \boldsymbol{\mu}^* (\zeta_x^* Q_{3i}^* + \zeta_y^* \delta_{3i})] \Psi_i$$

$$= \sum_\alpha \sum_i \nabla \cdot (f \boldsymbol{\mu}^* \zeta_\alpha^*) \psi_\alpha + \sum_i f \zeta_x^* \boldsymbol{\mu}^* \cdot \nabla Q_{3i}^* \Psi_i. \quad (5.36)$$

Upon substituting this result and eq. (5.30) into eq. (5.35), we get

$$\langle \tilde{T} f \zeta^*, \psi \rangle = \int \mathrm{d}\boldsymbol{r} \Big[- \sum_\alpha \nabla \cdot \{ f \zeta_\alpha^* (\boldsymbol{\mu}^* - \boldsymbol{r}) \} \psi_\alpha + \sum_\alpha (\boldsymbol{\mu}^* - \boldsymbol{r}) \cdot \nabla K_\alpha^* \delta \zeta_\alpha + \sum_\alpha \sum_{ij} f \zeta_\alpha^* \Big(\frac{\partial \boldsymbol{D}_{\alpha i}}{\partial p_j} \Big)^* \cdot \nabla p_i^* \Psi_j^{\mathrm{p}} - \sum_i f \zeta_x^* \boldsymbol{\mu}^* \cdot \nabla Q_{3i}^* \Psi_i^{\mathrm{p}} \Big]. \quad (5.37)$$

The first term between square brackets contributes a factor ∇f to $\tilde{T} f \zeta^*$. It drops out if and only if

$$\boldsymbol{\mu}^* - \boldsymbol{r} = 0. \quad (5.38)$$

As anticipated, the second term between square brackets (which is proportional to Ψ^{t}) drops out at the same time. Hence, eq. (5.38) determines the local fixed point. From eqs. (5.24) and (4.3), one finds that the functions S_1^*, S_2^* and S_3^* solving eq. (5.38), are those given by eq. (4.23). We have already seen that these are *indeed* fixed-point solutions! After elimination of Ψ^{p} by means of eq. (5.25), one arrives at the following expression for $\tilde{T} f \zeta^*$ at this local fixed point,

$$\sum_\alpha \tilde{T}_{\beta\alpha} f \zeta_\alpha^* = f \mu_0^{*-1} \sum_i \Big[\sum_\alpha \sum_j \zeta_\alpha^* \Big(\frac{\partial \boldsymbol{D}_{\alpha j}}{\partial p_i} \Big)^* \cdot \nabla p_j^* - \boldsymbol{\mu}^* \cdot \nabla Q_{3i}^* \Big] \eta_i^* \zeta_\beta^*. \quad (5.39)$$

Following HSL, we label the eigenfunctions f with a parameter $\boldsymbol{\rho}$ by writing

$$f_{\boldsymbol{\rho}}(\boldsymbol{r}) = \delta(\boldsymbol{r} - \boldsymbol{\rho}). \quad (5.40)$$

The corresponding eigenvalues $y_{\boldsymbol{\rho}}$ are according to eq. (5.34) given by

$$y_{\boldsymbol{\rho}} = \mu_0^{*-1} \sum_i \Big[\sum_\alpha \sum_j \zeta_\alpha^* \Big(\frac{\partial \boldsymbol{D}_{\alpha j}}{\partial p_i} \Big)^* \cdot \nabla p_j^* - \boldsymbol{\mu}^* \cdot \nabla Q_{3i}^* \Big] \eta_i^* \Big|_{\boldsymbol{r} = \boldsymbol{\rho}}. \quad (5.41)$$

Upon inserting the explicit expressions for the quantities appearing in this equation and using the expressions for the matrixelements of $\partial Q/\partial p$ given in appendix D of HSL, one finds after a straightforward, though tedious, calculation $y_{\rho} = 1$ for all $\boldsymbol{\rho}$. Consequently, the eigenvalue $y_T \equiv y_{\rho} = 1$ is infinitely degenerate. Since $\tilde{T}$ allows the eigenfunctions (5.40), the eigenvalue $y_T = 1$ does not only pertain to properties of the system as a whole, but also to critical properties of each local subsystem, as we anticipated by using the name local fixed point. Indeed, in view of the relation $2 - \alpha = d/y_T$, the eigenvalue $y_T = 1$ is in agreement with the well-known exact solution for infinite homogeneous Ising models.

6. Lattices with an oblique edge

One of the intriguing features of differential real space renormalization theory is the role played by the boundary conditions. First of all, while they are necessary to ensure the infinitesimal character of the transformation and force us to consider lattices with slowly varying interaction parameters, the boundary conditions should after all play no role in drawing conclusions concerning homogeneous systems, because one is interested then in local properties of the transformation instead of global ones. Secondly, the boundary conditions entail a very singular behaviour at the corners of the lattice, and one may wonder whether this is not in conflict with the assumption that the interaction parameters are slowly varying in space. In this section we will try to shed some light on these points by considering two seemingly similar lattice-shapes. It will be argued that for one of them the boundary conditions are too singular to allow simple (in a sense to be specified later) fixed-point solutions, while for the other lattice-shape a simple fixed-point solution is easily derived. Both lattice-shapes have no local fixed point, however.

We first consider a lattice to be called a pyramid (a square lattice of which the bottom row contains $L/a + 1$ spins, the row on top of it $L/a - 1$, etc.). The transformation at the right edge is depicted in fig. 6. In each cycle we introduce one extra new cross spin by means of a truncated inverse star-triangle transformation. In this step (a)$\rightarrow$(b), we have some freedom to choose these interactions, indicated by p_1' and p_2' in fig. 6b. In order that the transformation becomes infinitesimal, we have to require that the dotted interactions in triangles I and II are the same (up to order a/L). For the interactions q_1 in these triangles, this leads to the requirement (the interactions p_1'' and p_2'' are indicated in fig. 6b)

$$F(0, p_2', K_y) = F(p_1'', p_2'', K_y). \tag{6.1}$$

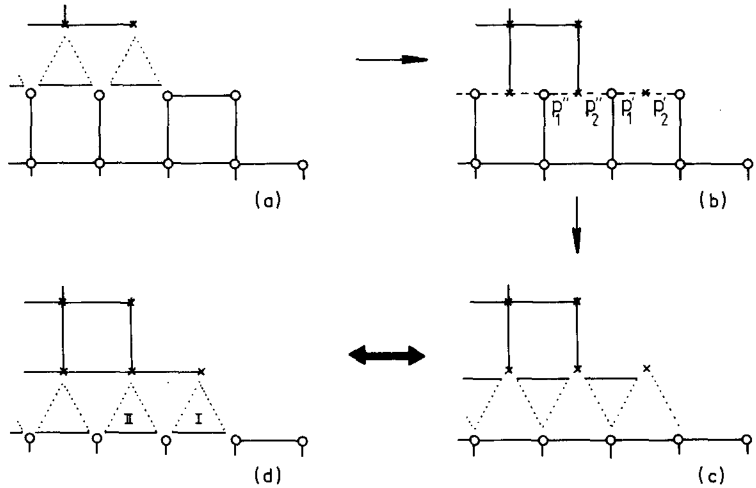

Fig. 6. Modified restructuring transformation for lattices with an oblique edge.

Eq. (6.1) is e.g. fulfilled if we take $p_2' = \infty$ provided that we are sure that this also implies $p_2'' \to \infty$. It is unclear what conditions have to be imposed on K_x and K_y, if we consider these as given functions, to ensure that this is the case. Nevertheless, if the search for fixed-point solutions is done in terms of p_1^* and p_2^*, one just has to restrict the analysis to functions p_2^* that diverge at the right boundary. Similar conclusions hold for the other interactions in the triangles I and II and at the left boundary, where p_1 has to diverge. As usual, the new lattice of cross spins has to be stretched isotropically to recover its original size. The transformation is therefore again described in the limit $a/L \to 0$ by equations (2.31)–(2.33), which now have to be solved in the domain $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $0 \leq y \leq \min\{\frac{1}{2} - x, \frac{1}{2} + x\}$.

As before, we will look for so-called simple fixed-point solutions, i.e. solutions of the type (4.10). As is clear from eq. (4.10), a function S_1^* of this type can not diverge at the right boundary (where $x + y = \frac{1}{2}$); consequently, S_2^* can not diverge either†. Therefore, all simple fixed-point solutions can only satisfy the boundary conditions

$$S_1^* = \infty; \quad S_2^* = S_1^{*-1} \text{ finite}, \quad \text{for } -x + y = \tfrac{1}{2}, \quad x < 0, \tag{6.2}$$

$$S_2^* = \infty; \quad S_1^* = S_2^{*-1} \text{ finite}, \quad \text{for } x + y = \tfrac{1}{2}, \quad x > 0. \tag{6.3}$$

At the bottom edge, we still have the boundary condition (4.8). The above boundary conditions are of a different type than those considered before: in contrast to all fixed points found so far by HSL, Knops and Hilhorst[3]), as well

† This is the reason that we did not discuss the case $K_y \to \infty$ or $K_y \to 0$ in analysing eq. (6.1).

as in section 4, only one function is infinite in eqs. (6.2), (6.3) at the oblique edge. As discussed in appendix C, it appears that there do not exist simple fixed-point solutions. However, if we enlarge the pyramid to the domain $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $0 \leq y \leq \frac{1}{2} - x$, to be called a triangle, there is just one oblique edge. In this case, the proper boundary conditions for simple solutions are, besides the one given in eq. (4.8),

$$S_1^* = \infty, \quad S_3^* = \infty, \quad S_2^* = 0, \quad \text{for } x = -\tfrac{1}{2}, \quad 0 < y < 1, \tag{6.4}$$

$$S_2^* = \infty, \quad S_1^* = S_3^{*-1} \text{ finite}, \quad \text{for } x + y = \tfrac{1}{2}, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}. \tag{6.5}$$

The following simple fixed-point solution satisfies these boundary conditions (see appendix C for a derivation)

$$S_1^* = \frac{(1+y)^{1/2}(3/2 + x - y)}{2y(\frac{1}{2} + x)^{1/2}}, \quad S_2^* = \frac{(1+y)^{1/2}(\frac{1}{2} + x)^{1/2}(\frac{1}{2} + x + y)}{2y(\frac{1}{2} - x - y)}, \tag{6.6}$$

with S_3^* given by eq. (4.3).

It should be noted that both lattice-shapes considered in this section have no local fixed-point solution, since the functions (4.23) do not obey the boundary condition at the oblique edge(s).

Though the question of the presence of a simple fixed-point solution is a poor criterium to judge the renormalization transformation, the above analysis demonstrates that the applicability of the present renormalization transformation is not restricted to semi-infinite lattices. It also shows that it may be useful to investigate different lattice shapes in searching for fixed-point solutions. Moreover, even though the precise conditions under which eq. (6.1) is fulfilled are unknown, it yields an example of a proper renormalization transformation for which local critical behaviour can not be studied due to the fact that there is no local fixed-point solution.

7. Concluding remarks

In this paper we have derived exact RG equations for the Ising model on a square lattice on the basis of a restructuring transformation invented by Hilhorst. The equations are more involved than those found in previous applications of differential RG theory, in that the flow equations contain two auxiliary parameters; to obtain explicit expressions for the renormalization flow, these parameters have to be determined by solving a partial differential equation. To evaluate the critical behaviour, the concept of local fixed point was introduced. The linearized flow around such a fixed point describes local properties of the system and can be associated with the critical properties of homogeneous systems.

Finally, we would like to comment further on the concept of local fixed point.

1. The local fixed point is determined by the RG equations irrespective of the shape of the lattice and the boundary conditions. For, one should realize a) that the relation $\Sigma_\alpha \zeta_\alpha \boldsymbol{D}_{\alpha i} = \mu_0 \boldsymbol{\mu} \eta_i$ (cf. eq. (5.23)) expresses a fundamental property of the transformation *in the bulk*; b) that according to the program set forth by HSL, the essential feature of the present formulation of differential real space renormalization theory is the comparison of two nearly identical lattices with a different lattice constant, so that the *densities* of degrees of freedom of the two lattices are slightly different; as a result, stretching terms of the type $\boldsymbol{r} \cdot \nabla K_\alpha$ arise in the flow equations. It is on the basis of these two properties that a local fixed point will in general be found from an equation of the type $\boldsymbol{\mu}^* - \boldsymbol{r} = 0$, without reference to any lattice shape or boundary conditions. This observation has three important consequences:

(i) The local fixed point (if it exists) can be determined by a short-cut, i.e. *without solving a partial differential equation.*

(ii) The local fixed point more or less prescribes what should be considered as the natural shape of the lattice, in that it will in general be compatible with the boundary conditions of *only one particular lattice shape.* In our case, the natural lattice shape turned out to be the semi-infinite strip.

(iii) The relation $\boldsymbol{\mu}^* - \boldsymbol{r} = 0$ indicates in our case what should be considered as the *right direction* in which the sequence of star-triangle transformations should move through the lattice. To see this, suppose we had started the restructuring transformation at the bottom row of the semi-infinite strip (in the way indicated in fig. 4) and had then advanced upwards through the strip. Let us denote the analogues of the vectors $\boldsymbol{D}$ and $\boldsymbol{M}$ in the RG equations describing this up-transformation by $\boldsymbol{D}^u$ and $\boldsymbol{M}^u$. Because the up-transformation moves in the $+y$ direction and the down-transformation in the $-y$ direction, we have $\boldsymbol{D}^u = (\boldsymbol{D} \cdot \boldsymbol{e}_x)\boldsymbol{e}_x - (\boldsymbol{D} \cdot \boldsymbol{e}_y)\boldsymbol{e}_y$; an analogous result holds for $\boldsymbol{M}^u$. The stretching term in this case, however, should be the same as before. From the above expression for $\boldsymbol{D}^u$, it follows that $\boldsymbol{\mu}^u = (\boldsymbol{\mu} \cdot \boldsymbol{e}_x)\boldsymbol{e}_x - (\boldsymbol{\mu} \cdot \boldsymbol{e}_y)\boldsymbol{e}_y$, so that $\boldsymbol{e}_y \cdot (\boldsymbol{\mu}^{u*} - \boldsymbol{r}) = -(S_1^*/(S_1^* + S_2^*) + y) < 0$ (cf. eq. (5.24)). The conclusion therefore is, that the up-transformation has no local fixed point. To understand this result, note that the terms $\Sigma_i \boldsymbol{D}_{\alpha i} \cdot \nabla p_i$ and $-\boldsymbol{r} \cdot \nabla K_\alpha$ should in general have opposite effects for a proper transformation with given boundary conditions: one of them should have the tendency to increase the interaction K_α under renormalization, the other to decrease K_α; the two terms should just balance at the fixed point. Apparently, in an up-transformation this is not the case.

2. For an arbitrary differential renormalization transformation with flow equations of the type obtained by HSL or of our equations (2.31)–(2.33), the

local fixed point should obey an equation of the form $\boldsymbol{\mu}^* - \boldsymbol{r} = 0$. Hence, in d dimensions, the local fixed point is in general determined by $d+1$ equations: since $\boldsymbol{\mu}^*(p_1^*, p_2^*, \ldots)$ does not depend on $\boldsymbol{r}$, the equation $\boldsymbol{\mu}^* - \boldsymbol{r} = 0$ is equivalent to d independent equations†, and the requirement that p_1, p_2 etc. lie in the critical subspace yields one extra equation, independent from the previous ones. Consequently, a local fixed-point solution of a renormalization transformation can only be found if there are at least $d+1$ parameters to solve these equations‡. Consequently, the formulation of a proper renormalization scheme with isotropic dilation in terms of the interaction parameters K_x and K_y alone, seems to be ruled out.

3. At the local fixed point, one finds in general one unique eigenvalue y_T. As was stressed in particular by Van Leeuwen[7]), one can then, according to the renormalization picture, construct the exact nonlinear scaling fields in the interaction space considered. In terms of these fields, corrections to scaling are absent. Hence, if differential real space renormalization theory can be applied to still other models, we expect to find local fixed points only for those models that exhibit merely trivial (analytic) corrections to scaling.

4. The linearized flow in the temperature direction around non-local fixed points is more difficult to analyse with the present RG equations than with those of HSL. Without a detailed analysis it is clear however, that the eigenvalue equation for these fixed points is in the case of HSL of the general form§

$$B^*f + (\boldsymbol{\mu}^* - \boldsymbol{r}) \cdot \nabla f = y_T f.$$

Since the eigenfunctions f will usually be non-zero everywhere, they will describe properties of the system as a whole. It turns out that one of the

† We could get less than d equations if it would be possible to obtain a RG transformation by comparing lattices with lattice constants that are different in some directions, but not in all. This would result in non-isotropic stretching. It seems unlikely, however, that such a procedure is consistent. For, if we do not stretch the semi-infinite strip in the y-direction, we expect ξ_y, the correlationlength in the y-direction, to remain unchanged under renormalization. But $\xi_y(K') = \xi_y(K)$ implies $K' = K$, in disagreement with the RG picture.

‡ It should be emphasized that the problem is to obtain solutions of at least $d+1$ independent ordinary equations. In our analysis, we considered K_x and K_y as the two completely independent functions, and viewed p_1 and p_2 as parameters depending on K_x and K_y; p_1 and p_2 are then obtained from one ordinary equation, eq. (2.31), and one differential equation, eq. (2.32). The solution of the latter equation is not of the form $f(p_1(\boldsymbol{r}), p_2(\boldsymbol{r}), K_x(\boldsymbol{r}), K_y(\boldsymbol{r})) = 0$, but yields a functional relation, that can symbolically be written as $h([p_1(\boldsymbol{r})], [p_2(\boldsymbol{r})], [K_x(\boldsymbol{r})], [K_y(\boldsymbol{r})]) = 0$. Thus we indeed had 3 functions (e.g. $p_1(\boldsymbol{r})$, $p_2(\boldsymbol{r})$ and $K_y(\boldsymbol{r})$) that, though not fully independent, were not restricted further by an ordinary equation.

§ In our case, the eigenvalue equation does not appear to be of the above form due to the second term between square brackets in eq. (5.37), which is proportional to Ψ'. Nevertheless, since this one is also proportional to $\boldsymbol{\mu}^* - \boldsymbol{r}$, an analysis, similar to the one given above, seems to apply.

components of $\mu^* - r$ vanishes at the boundaries for all non-local fixed-point solutions found by Knops and Hilhorst[3]). This might lead to divergent behaviour of f at the edges; if f is divergent, however, it does not describe a small deviation from criticality any more, so that the use of the linearized flow equations becomes questionable. Maybe such behaviour should have to be interpreted in the light of recent results of Hilhorst and Van Leeuwen[17]), who study Ising systems of which the interactions close to the boundary are weaker than those in the bulk. If the deviations from the bulk values diverge as $1/\Delta^z$, where Δ is the distance from the boundary, they find a non-universal decay of the boundary spin–spin correlation function when the bulk is critical.

In this connection we wish to recall some analogous features of the differential RG analysis of the linear Ising chain with nearest-neighbour interactions[11]). In this case, the transformation is formulated in terms of two parameters (cf. remark 2) as a result of which a whole class of renormalization transformations is obtained. Although all transformations have the same zero-temperature fixed-point, only one of them allows the study of local critical behaviour. The spectrum of eigenvalues of other "nonlocal" transformations is unbounded, but the corresponding eigenfunctions can not be interpreted as small deviations from criticality, since they are all divergent at some point. Apparently, there is no clear interpretation of these "non-local" transformations.

Acknowledgements

I am indebted to Dr. H.J. Hilhorst and Dr. D. Bedeaux for many illuminating discussions and for their encouragement. I also profited by a stimulating discussion with Professor J.M.J. van Leeuwen and Dr. B. Nienhuis, by advice from G.R.W. Quispel and Professor L.A. Peletier on strategies to solve the fixed-point equations, as well as by the comments of Professor P.W. Kasteleyn on a preliminary version of the manuscript.

Appendix A

In this appendix we derive eq. (3.10). One has (cf. eqs. (4.11) and (4.12) of HSL)

$$\frac{\partial \bar{p}_i}{\partial p_l} = -\frac{\tanh 2\bar{q}_l}{\tanh 2p_l}\bar{Q}_{il}^{-1} = -Q_{il}\frac{\tanh 2\bar{p}_i}{\tanh 2q_i}, \quad i, l = 1, 2, 3, \tag{A.1}$$

$$\frac{\partial p_i}{\partial \bar{p}_l} = -\frac{\tanh 2q_l}{\tanh 2\bar{p}_l}Q_{il}^{-1} = -\bar{Q}_{il}\frac{\tanh 2p_i}{\tanh 2\bar{q}_i}, \quad i, l = 1, 2, 3. \tag{A.2}$$

These equations imply for $|Q| := \det Q$

$$|Q| = |\bar{Q}|^{-1} \prod_{i=1}^{3\prime} \frac{\tanh 2q_i \tanh 2\bar{q}_i}{\tanh 2p_i \tanh 2\bar{p}_i}. \tag{A.3}$$

With the aid of eq. (A.2), eq. (2.23) can be written as

$$-\sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{2} (-1)^j Q_{ji} Q_{(3-j)3} Q_{ik}^{-1} \frac{\tanh 2q_k}{\tanh 2\bar{p}_k} (\boldsymbol{b}_i + \boldsymbol{b}_j) \cdot \nabla \bar{p}_k = 0. \tag{A.4}$$

On the basis of Cramer's rule for the matrixelements of the inverse of a matrix, we have

$$\sum_{j=1}^{2} (-1)^j Q_{ji} Q_{(3-j)3} = \begin{cases} 0, & \text{for } i = 3, \\ -(-1)^i Q_{(3-i)3}^{-1} |Q|, & \text{for } i = 1, 2. \end{cases} \tag{A.5}$$

We first rewrite the term proportional to $\boldsymbol{b}_i$ in eq. (A.4); upon using eq. (A.5) and next eq. (A.2), we get

$$\begin{aligned} &-\sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{2} (-1)^j Q_{ji} Q_{(3-j)3} Q_{ik}^{-1} \frac{\tanh 2q_k}{\tanh 2\bar{p}_k} \boldsymbol{b}_i \cdot \nabla \bar{p}_k \\ &= \sum_{k=1}^{3} \sum_{i=1}^{2} (-1)^i Q_{(3-i)3}^{-1} Q_{ik}^{-1} |Q| \frac{\tanh 2q_k}{\tanh 2\bar{p}_k} \boldsymbol{b}_i \cdot \nabla \bar{p}_k, \\ &= \sum_{k=1}^{3} \sum_{i=1}^{2} (-1)^i \bar{Q}_{(3-i)3} \bar{Q}_{ik} |Q| \frac{\tanh 2p_i \tanh 2p_{3-i} \tanh 2\bar{p}_3}{\tanh 2\bar{q}_i \tanh 2\bar{q}_{3-i} \tanh 2q_3} \boldsymbol{b}_i \cdot \nabla \bar{p}_k \\ &= J \sum_{k=1}^{3} \sum_{i=1}^{2} (-1)^i \bar{Q}_{(3-i)3} \bar{Q}_{ik} \boldsymbol{b}_i \cdot \nabla \bar{p}_k, \end{aligned} \tag{A.6}$$

where

$$J := |Q| \frac{\tanh 2p_1 \tanh 2p_2 \tanh 2\bar{p}_3}{\tanh 2\bar{q}_1 \tanh 2\bar{q}_2 \tanh 2q_3}. \tag{A.7}$$

For the term proportional to $\boldsymbol{b}_j$ in (A.4), we similarly get (in the second step we use eq. (A.1), in the third eq. (A.5) for the barred variables and in the fourth step eq. (A.3))

$$\begin{aligned} &-\sum_{k=1}^{3} \sum_{i=1}^{3} \sum_{j=1}^{2} (-1)^j Q_{ji} Q_{(3-j)3} Q_{ik}^{-1} \frac{\tanh 2q_k}{\tanh 2\bar{p}_k} \boldsymbol{b}_j \cdot \nabla \bar{p}_k \\ &= -\sum_{k=1}^{3} \sum_{j=1}^{2} (-1)^j \delta_{jk} Q_{(3-j)3} \frac{\tanh 2q_k}{\tanh 2\bar{p}_k} \boldsymbol{b}_j \cdot \nabla \bar{p}_k, \\ &= -\sum_{j=1}^{2} (-1)^j \bar{Q}_{(3-j)3}^{-1} \frac{\tanh 2q_j \tanh 2q_{3-j} \tanh 2\bar{q}_3}{\tanh 2\bar{p}_j \tanh 2\bar{p}_{3-j} \tanh 2p_3} \boldsymbol{b}_j \cdot \nabla \bar{p}_j, \\ &= |\bar{Q}|^{-1} \frac{\tanh 2q_1 \tanh 2q_2 \tanh 2\bar{q}_3}{\tanh 2\bar{p}_1 \tanh 2\bar{p}_2 \tanh 2p_3} \sum_{j=1}^{3} \sum_{i=1}^{2} (-1)^i \bar{Q}_{ij} \bar{Q}_{(3-i)3} \boldsymbol{b}_j \cdot \nabla \bar{p}_j, \end{aligned}$$

$$= J \sum_{j=1}^{3} \sum_{i=1}^{2} (-1)^i \bar{Q}_{ij} \bar{Q}_{(3-i)3} \boldsymbol{b}_j \cdot \nabla \bar{p}_j. \tag{A.8}$$

If we substitute eqs. (A.6) and (A.8) into eq. (A.4) and divide by J, we find

$$\sum_{i=1}^{3} \sum_{j=1}^{2} (-1)^j \bar{Q}_{ji} \bar{Q}_{(3-j)3} (\boldsymbol{b}_i + \boldsymbol{b}_j) \cdot \nabla \bar{p}_i = 0, \tag{A.9}$$

which is eq. (2.32) with the variables p_i replaced by $\bar{p}_i$.

Appendix B

In this appendix eqs. (4.4) and (4.5) are derived. To obtain eq. (4.5), we substitute eqs. (5.6) and (5.7) into eq. (2.18); if we employ the notation (4.2) and (5.10), the resulting equation reads

$$(C_1 + C_2 C_3) \frac{\partial p_1}{\partial y} - (C_2 + C_1 C_3) \frac{\partial p_2}{\partial y} + C_1 \frac{\partial p_1}{\partial x} + C_2 \frac{\partial p_2}{\partial x} - C_1 C_2 \frac{\partial p_3}{\partial x} = 0. \tag{B.1}$$

Using the result (5.12), this may be rewritten as

$$S_2 S_3 \frac{\partial S_1}{\partial y} - S_1 S_3 \frac{\partial S_2}{\partial y} + \frac{\partial S_1}{\partial x} + \frac{\partial S_2}{\partial x} - (S_1 S_2 - 1) \frac{\partial S_3}{\partial x} = 0. \tag{B.2}$$

The terms containing derivatives with respect to x can be combined using the fact that $S_1 + S_2 + S_3 = S_1 S_2 S_3$ (this is just eq. (4.3)). One gets

$$S_2 S_3 \frac{\partial S_1}{\partial y} - S_1 S_3 \frac{\partial S_2}{\partial y} + S_3 \frac{\partial S_1 S_2}{\partial x} = 0. \tag{B.3}$$

This equation reduces to eq. (4.5) after dividing it by $S_1 S_2 S_3$.

At the fixed-point, the right-hand side of eq. (2.26) has to be zero; by eliminating K_x from the resulting equation and using eqs. (5.6) and (5.7), we obtain

$$\frac{1}{2} C_2^* \frac{\partial p_1^*}{\partial x} - \frac{1}{2} C_1^* \frac{\partial p_2^*}{\partial x} + \frac{\partial K_y^*}{\partial y} + (S_1^* S_2^* + 1)\left(x \frac{\partial K_y^*}{\partial x} + y \frac{\partial K_y^*}{\partial y} \right) = 0. \tag{B.4}$$

From eqs. (5.11) and (5.12) it is easy to show that

$$\frac{1}{C_1^*} \nabla p_1^* + \frac{1}{C_2^*} \nabla p_2^* + \frac{1}{C_3^*} \nabla p_3^* = 0. \tag{B.5}$$

After dividing eq. (B.4) by $C_1^* C_2^*$ and using eqs. (5.12) and (B.5), one obtains after rearranging terms

$$(1+S_1^*S_2^*)\left[(1+S_2^{*2})\left\{\left(x-\frac{1}{2}\right)\frac{\partial S_1^*}{\partial x}+y\frac{\partial S_1^*}{\partial y}\right\}+(1+S_1^{*2})\left\{\left(x+\frac{1}{2}\right)\frac{\partial S_2^*}{\partial x}+y\frac{\partial S_2^*}{\partial y}\right\}\right]$$
$$+(1+S_2^{*2})\left(\frac{\partial S_1^*}{\partial x}+\frac{\partial S_2^*}{\partial y}\right)-(1+S_1^{*2})\left(\frac{\partial S_2^*}{\partial x}-\frac{\partial S_1^*}{\partial y}\right)=0. \tag{B.6}$$

From eq. (4.5) it also follows that

$$(1+S_2^{*2})\left(\frac{\partial S_1^*}{\partial x}+\frac{\partial S_2^*}{\partial y}\right)-(1+S_1^{*2})\left(\frac{\partial S_2^*}{\partial x}-\frac{\partial S_1^*}{\partial y}\right)$$
$$=(1+S_1^*S_2^*)\left(\frac{\partial S_1^*}{\partial x}+\frac{\partial S_1^*}{\partial y}-\frac{\partial S_2^*}{\partial x}+\frac{\partial S_2^*}{\partial y}\right). \tag{B.7}$$

By combining eqs. (B.6) and (B.7), one obtains eq. (4.4).

Appendix C

In this appendix, we first show how the fixed-point solution (6.6) was obtained and then argue why the ansatz (4.10) appears to break down for the pyramid.

In view of the boundary conditions (6.4), (6.5) and (4.8), $m(y)$ must diverge at the bottom edge, $n(x)$ at the left boundary and $M_2(x+y)$ at the oblique edge. We first consider the left bottom corner, $x=-\frac{1}{2}$, $y=0$. Since $M_1(x-y)$ can not be singular along the line with equation $y=\frac{1}{2}+x$, M_1 should approach a finite value at this corner; all other functions can in principle diverge, and we will therefore assume $m(y)\sim y^{-\mu}$, $n(x)\sim(\frac{1}{2}+x)^{-\nu}$, $M_2(x+y)\sim(\frac{1}{2}+x+y)^{-\lambda}$. If we substitute this anzatz in eq. (4.4) and require that the most diverging terms cancel each other in this equation in the three respective limits, $\lim x\downarrow -\frac{1}{2}\lim y\downarrow 0$, $\lim y\downarrow 0\lim x\downarrow -\frac{1}{2}$ and $\lim y\downarrow 0$ $(x+\frac{1}{2})/y$ constant, one finds that this is only possible if

$$S_1^*\simeq c_1y^{-1}(\tfrac{1}{2}+x)^{-1/2},\quad S_2^*\simeq c_1(\tfrac{1}{2}+x)^{1/2}y^{-1},\quad y,(\tfrac{1}{2}+x)\ll 1. \tag{C.1}$$

Here c_1 is some unknown constant. Similarly, one finds

$$S_1^*\simeq c_2y^{-1},\quad S_2^*\simeq c_2y^{-1}(\tfrac{1}{2}-x-y)^{-1},\quad y,(\tfrac{1}{2}-x)\ll 1, \tag{C.2}$$

$$S_1^*\simeq\frac{c_3(3/2+x-y)(\frac{1}{2}+x)^{-1/2}}{\sqrt{2}},\quad S_2^*\simeq\frac{c_3^{-1}(\frac{1}{2}+x)(\frac{1}{2}-x-y)^{-1}}{\sqrt{2}},$$
$$(1-y),\quad (\tfrac{1}{2}+x)\ll 1, \tag{C.3}$$

where c_2 and c_3 are other unknown constants. Obviously $m(y)\sim y^{-1}$ at the bottom edge, and we therefore write, as in section 4, $m(y)=m_{ns}(y)/y$. Upon substituting this in eq. (4.4), this equation becomes of the form

$$y^{-3}U(x, y) + y^{-2}W(x, y) + \mathcal{O}(y^{-1}) = 0, \tag{C.4}$$

where $U(x, y)$ and $W(x, y)$ are known functions of $m_{ns}(y)$, $n(x)$, $M_1(x-y)$ and $M_2(x+y)$. Clearly, it is necessary to have $U(x, 0) = 0$ in eq. (C.4). This requirement leads to eq. (4.11). One may check that the solutions (4.17) of this equation are incompatible with the asymptotic behaviour (C.1) and (C.2). However, the solutions (4.16) are allowed; from eqs. (4.13), (4.14) and (4.16) one thus finds that

$$\frac{M_1(x)n^2(x)}{M_2(x)} = \frac{\frac{1}{2} - x}{\frac{1}{2} + x}. \tag{C.5}$$

Next, we require that the terms of order y^{-2} cancel in eq. (C.4). This is the case provided that

$$\left.\frac{\partial U(x, y)}{\partial y}\right|_{y=0} + W(x, 0) = 0. \tag{C.6}$$

With the aid of the explicit expressions for U and W, this equation becomes after eliminating $n(x)$ by means of eq. (C.5)

$$\frac{m'_{ns}(0)}{m_{ns}(0)} - \{m^2_{ns}(0)M_1(x)M_2(x)\}^{-1} + \left(-\frac{1}{2} - 2x\right)\frac{\mathrm{d}\ln M_1(x)}{\mathrm{d}x} + \left(\frac{1}{2} - 2x\right)\frac{\mathrm{d}\ln M_2(x)}{\mathrm{d}x}$$
$$-\frac{1}{2}\left(\frac{1}{2} + x\right)\left(\frac{1}{2} - x\right)\left[\left(\frac{\mathrm{d}\ln M_1(x)M_2(x)}{\mathrm{d}x}\right)^2 - 2\frac{\mathrm{d}^2\ln M_1(x)M_2(x)}{\mathrm{d}x^2}\right] = 0. \tag{C.7}$$

Of course, $m_{ns}(0)$ and the derivative $m'_{ns}(0)$ are not known. Eq. (C.7) only needs to hold for $-\frac{1}{2} \leq x < \frac{1}{2}$. However, neither $M_1(x)$ nor $M_2(x)$ is singular at $x = -\frac{1}{2}$; let us therefore assume that M_1 and M_2 also obey this equation for $-3/2 < x \leq -\frac{1}{2}$. For $x \downarrow -3/2$, M_1 behaves as $M_1 \sim 3/2 + x$, as eq. (C.3) shows. With this asymptotic behaviour, the terms between square brackets in eq. (C.7) diverge as $(3/2 + x)^{-2}$, and these terms can be cancelled by the one between curly brackets provided that also $M_2 \approx 3/2 + x$ and that $m_{ns}(0)$ has a proper value. At this stage, one soon realises that

$$M_1(x) = 3/2 + x, \quad M_2(x) = \frac{3/2 + x}{\frac{1}{2} - x}, \tag{C.8}$$

is an exact solution of eq. (C.7) that satisfies the asymptotic behaviour (C.2), provided that $m_{ns}(0) = \frac{1}{2}$ and $m'_{ns}(0) = \frac{1}{4}$. The function $m(y)$ now follows from an analysis at the oblique edge: we substitute the expression for $M_2(x + y)$ into eq. (4.4), and require, as usual, that the most diverging terms (which are of order $(\frac{1}{2} - x - y)^{-2}$) cancel. This turns out to be the case provided that

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln\left(1 + m^2(y)n^2\left(\frac{1}{2} - y\right)M_1^2\left(\frac{1}{2} - 2y\right)\right) = -\frac{M_1(\frac{1}{2} - 2y)n^2(\frac{1}{2} - y)}{y}. \tag{C.9}$$

After substitution of the expressions for M_1 and n, we obtain by an elementary integration

$$m(y) = (1+y)^{1/2}/2y. \tag{C.10}$$

On the basis of eqs. (C.5), (C.8) and (C.10), one obtains the functions (6.7), which indeed turn out to be fixed-point solutions of eq. (4.4).

In analyzing the pyramid with the ansatz (4.10), we assume, as before, that M_2 diverges as $(\frac{1}{2}-x-y)^{-\gamma}$. With this assumption, the most divergent terms at the right edge are of order $(\frac{1}{2}-x-y)^{-2\gamma}$ and $(\frac{1}{2}-x-y)^{-\gamma-1}$. Obviously these terms can only cancel if $\gamma = 1$, and hence, if eq. (C.9) is obeyed. Eq. (C.9) is, however, inconsistent if $n(0)$ is finite. For the right-hand side of eq. (C.9) diverges as $(\frac{1}{2}-y)^{-1}$ in the limit $y \uparrow \frac{1}{2}$, since $M_1(\frac{1}{2}-2y)$ will diverge as $(\frac{1}{2}-y)^{-1}$ in this limit in view of the boundary conditions at the left edge. According to eq. (C.9), the argument of the logarithm should then have to vanish as $\frac{1}{2}-y$, but this is clearly impossible. The conclusion therefore is that no solutions of the type (4.10) exist if we assume that M_2 diverges as $(\frac{1}{2}-x-y)^{-\gamma}$ and that $n(0)$ is finite.

Note added in proof

Attempts to construct differential real space renormalization (DRSR) schemes for which the lattices $\mathscr{L}$ and $\mathscr{L}'$ have a sublattice in common, are beset with problems even more severe than those which plague checkerboard transformations[18]). For in such a scheme the correlation of two spins of the common sublattice remains unchanged under renormalization, while in DRSR their distance measured in units of the lattice spacing is the same on $\mathscr{L}$ and $\mathscr{L}'$. Essentially, this implies that the correlation length does not scale properly. In this connection, it should be noted that the equations studied by Jezewski[19]) are indeed not proper RG equations.

References

1) H.J. Hilhorst, M. Schick and J.M.J. van Leeuwen, Phys. Rev. Lett. **40** (1978) 1605.
2) H.J. Hilhorst, M. Schick and J.M.J. van Leeuwen, Phys. Rev. **B19** (1979) 2749.
3) H.J.F. Knops and H.J. Hilhorst, Phys. Rev. **B19** (1979) 3689.
4) D. Ray, Phys. Rev. **B24** (1981) 473.
5) B. Payandeh and J.M.J. Van Leeuwen, Physica **106A** (1981) 589.
6) H.J. Hilhorst and J.M.J. Van Leeuwen, Physica **106A** (1981) 301.
7) J.M.J. van Leeuwen, in Perspectives in Statistical Physics, H.J. Raveché, ed. (North-Holland, Amsterdam, 1981).
8) Y. Yamazaki and H.J. Hilhorst, Phys. Lett. **70A** (1979) 329.
9) Y. Yamazaki, H.J. Hilhorst and G. Meissner, Z. Physik **B35** (1979) 333.

10) Y. Yamazaki, H.J. Hilhorst and G. Meissner, J. Stat. Phys. **23** (1980) 609.
11) W. van Saarloos, J.M.J. van Leeuwen and A.L. Stella, Physica **97A** (1979) 319.
12) R. Dekeyser and A. Stella, J. Stat. Phys. **23** (1980) 587.
13) A. Stella and R. Dekeyser, J. Stat. Phys. **25** (1981) 443.
14) H.J. Hilhorst, private communication, April 1978.
15) R.J. Baxter and I.G. Enting, J. Phys. **A11** (1978) 2463.
16) I. Syozi, in Phase Transitions and Critical Phenomena, vol. I, C. Domb and M.S. Green, eds. (Academic Press, New York, 1972).
17) H.J. Hilhorst and J.M.J. van Leeuwen, Phys. Rev. Lett. **47** (1981) 1188.
18) J.M.J. van Leeuwen, in: Fund. Probl. Stat. Mech. III (North-Holland, Amsterdam, 1975).
19) W. Jezewski, Phys. Rev. **B24** (1981) 3984.